

Finsler connection preserving the two-vector angle under the indicatrix-inhomogeneous treatment

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Abstract

The Finsler spaces in which the tangent Riemannian spaces are conformally flat prove to be characterized by the condition that the indicatrix is a space of constant curvature. In such spaces the Finslerian normalized two-vector angle can be explicated from the respective two-vector angle of the associated Riemannian space. Therefore the way is opening to propose explicitly the connection preserving the angle even at the indicatrix-inhomogeneous level, that is, when the indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ is permitted to be an arbitrary smooth function of the indicatrix position point x . The connection obtained is metrical with the deflection part which is proportional to the gradient of the function $H(x)$ entering the equality $\mathcal{C}_{\text{Ind.}} \equiv H^2$. Also the connection is covariant-constant. When the transitivity of covariant derivative is used, from the commutators of covariant derivatives the associated curvature tensor is found. Various useful representations have been developed. The Finsleroid space has been explicitly outlined.

Motivation and Introduction

In the Finsler space the tangent bundle TM over the base manifold M is geometrized by means of the Finsler metric function $F(x, y)$, such that at each point $x \in M$ the tangent vectors $y \in T_x M$ are used, where $T_x M$ is the tangent space supported by the x .

The embedded position of the indicatrix $\mathcal{I}_x \subset T_x M$ in the tangent Riemannian space $\mathcal{R}_{\{x\}} = \{T_x M, g_{\{x\}}(y)\}$ (where $g_{\{x\}}(y)$ denotes the Finslerian metric tensor with x considered fixed and y used as being the variable) induces the Riemannian metric on the indicatrix through the well-known method (see, e.g., Section 5.8 in [1]) and in this sense makes the indicatrix a Riemannian space. Therefore, the geodesics can be introduced on the indicatrix by applying the conventional Riemannian methods.

In any (sufficiently smooth) Finsler space the two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ can locally be determined with the help of the indicatrix geodesic arc, which provokes the important question whether the Finsler geometry can be profoundly settled down by developing and applying the connection which preserves the angle.

In general, the angle $\alpha_{\{x\}}(y_1, y_2)$ is complicated and cannot be determined in an explicit form, except for rare Finsler metric functions. The lucky example is given by the Finsler space \mathcal{F}^N which is characterized by the condition that the indicatrix is a space of constant curvature. In the space \mathcal{F}^N the angle $\alpha_{\{x\}}(y_1, y_2)$ can be found in the explicit and simple form. Namely, it is possible to prove that (under attractive conditions) in any dimension $N \geq 3$, the tangent Riemannian space $\mathcal{R}_{\{x\}}$ is conformally flat if and only if the indicatrix is a space of constant curvature. The respective transformation $y = \mathbf{C}(x, \bar{y})$ is positively homogeneous of the degree which we shall denote by H . The remarkable equality $\mathcal{C}_{\text{Ind.}} \equiv H^2$ arises, where $\mathcal{C}_{\text{Ind.}} = \mathcal{C}_{\text{Ind.}}(x)$ denotes the value of the curvature of the indicatrix $\mathcal{I}_x \subset T_x M$.

Under this transformation $y = \mathbf{C}(x, \bar{y})$ each tangent Riemannian space $\mathcal{R}_{\{x\}}$ is conformally changed to become a Euclidean space $\mathcal{E}_{\{x\}}$. The distribution of the last spaces $\mathcal{E}_{\{x\}}$ over the base manifold M composes the *associated Riemannian space*, which we denote by $\mathcal{R}^N = (M, S)$, where $S = \sqrt{a_{mn}(x)} y^m y^n$ is the Riemannian metric constructed from the metric tensor $a_{mn}(x)$ of the space $\mathcal{E}_{\{x\}}$. We are entitled to induce the angle $\alpha_{\{x\}}^{\text{Riem}}$ conventionally defined in the Riemannian space \mathcal{R}^N into the Finsler space \mathcal{F}^N , obtaining simply $\alpha_{\{x\}}(y_1, y_2) = (1/H(x)) \alpha_{\{x\}}^{\text{Riem}}(\bar{y}_1, \bar{y}_2)$.

To explicate the coefficients N^m_n of nonlinear connection from the Finsler angle $\alpha = \alpha_{\{x\}}(y_1, y_2)$, we should successfully propose the preservation equation. The nearest possibility is to formulate the equation $d_i \alpha = 0$ in accordance with the formulas (I.1.12) and (I.1.15), applying the separable operator d_i indicated in (I.1.11).

This possibility has been realized in the preceding work [10,11]. Namely, in that work the separable preservation equation $d_i \alpha = 0$ has been solved in the Finsler space \mathcal{F}^N under the assumption that $\mathcal{C}_{\text{Ind.}} = \text{const}$, and whence $H = \text{const}$. The explicit coefficients N^m_n have been obtained.

In general the indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ may depend on the points $x \in M$ which support the indicatrix. We call the space \mathcal{F}^N indicatrix-homogeneous, if the value is a constant, whence $H = \text{const}$. If the dependence $\mathcal{C}_{\text{Ind.}} = \mathcal{C}_{\text{Ind.}}(x)$ does hold, we say that the space \mathcal{F}^N is indicatrix-inhomogeneous, in which case $H_i \neq 0$, where $H_i = \partial H / \partial x^i$. The representations obtained in the previous work [10,11] are the $(H_i \rightarrow 0)$ -limits of their generalized counterparts developed in the present study.

It appears that in general the angle preservation equation formulated in the separable way does not permit any solution for the coefficients N^m_n .

This conclusion can be drawn from the implications which are derivable by the help of the coincidence-limit method (see Section 3.2 in [12]) which extracts the information from behavior of Riemannian geodesics. To this end we should use the distance function $E = E(x, y_1, y_2)$ with $E = (1/2)\alpha^2$. Namely, evaluating various partial derivatives of the function with respect to y_1 and y_2 and finding the coincidence limits when $y_2 \rightarrow y_1$, we can obtain a valuable information on the derivatives of the Finsler metric tensor of the Finsler space. Performing the required evaluations on the level of the second-order partial derivatives $\partial^2/\partial y_1^m \partial y_2^n$, and, then, applying the operation $y_2 \rightarrow y_1$ to the resultant expressions, it is possible to arrive at the following general conclusion: *In any Finsler space the vanishing assumption $d_i\alpha = 0$ of the separable type entails the equality*

$$\mathcal{D}_i h_{mn} = \frac{2}{F} h_{mn} d_i F.$$

If we additionally postulate $d_i F = 0$, we obtain $\mathcal{D}_i h_{mn} = 0$ and, therefore, the metricity $\mathcal{D}_i g_{mn} = 0$ which is formulated with the covariant derivative \mathcal{D} arisen from the deflectionless connection. We can apply the derivative $\partial^2/\partial y^l \partial y^h$ to the equality $\mathcal{D}_i g_{mn} = 0$, which leads after simple evaluations to the vanishing $\mathcal{D}_i S_n^k{}_{jm} = 0$. Here, the $S_n^k{}_{jm}$ is the tensor which describes the curvature of indicatrix (see Section 5.8 in [1]).

Clearly, the vanishing $\mathcal{D}_i S_n^k{}_{jm} = 0$ can be realized in but rare particular cases of the Finsler space. The vanishing is realized in the indicatrix-homogeneous case of the space \mathcal{F}^N , and cannot be realized in the indicatrix-inhomogeneous space \mathcal{F}^N .

Therefore, the account for the dependence $H = H(x)$ in the Finsler space \mathcal{F}^N is neither straightforward nor trivial task.

These important (and rather unexpected?) implications enforce us to look for more capable ideas to formulate the preservation of angle. The attractive idea is to substitute the normalized angle $\alpha_{\{x\}}^{\{H(x)\}}(y_1, y_2) = H(x)\alpha_{\{x\}}(y_1, y_2)$ (see (I.1.26)) with the initial angle $\alpha_{\{x\}}(y_1, y_2)$ in the separable preservation law, according to (I.1.27). The law obtained is of the recurrent-type (I.1.28), namely $d_i\alpha + (1/H)H_i\alpha = 0$. It appears that this preservation is reconciled with the indicatrix-inhomogeneous Finsler space \mathcal{F}^N at any scalar $H = H(x)$. The reason thereto is the following assertion obtainable by the help of the coincidence-limit method: *In any Finsler space the vanishing assumption $d_i\alpha + (1/H)H_i\alpha = 0$ entails the equality*

$$\mathcal{D}_i h_{mn} = \frac{2}{F} h_{mn} d_i F - \frac{2}{H} H_i h_{mn}.$$

In these patterns the vanishing $d_i F = 0$ yields the equality $\mathcal{D}_i g_{mn} = -(2/H)H_i h_{mn}$, which entails the extension of the previous vanishing $\mathcal{D}_i S_n^k{}_{jm} = 0$ such that the right-hand part of the extension is just the expression which is obtained when the characteristic representation of the tensor $S_n^k{}_{jm}$ of the space \mathcal{F}^N under study is inserted under the action of the covariant derivative \mathcal{D}_i .

Thus, the recurrent-type equation $d_i\alpha + (1/H)H_i\alpha = 0$ of preservation of the angle is reconciled with the indicatrix-inhomogeneous Finsler space \mathcal{F}^N at any scalar $H = H(x)$ (see Proposition I.1.2 in Section I.1), and therefore is accepted in the present work to apply. We solve the equation with respect to the coefficients $N^m{}_n(x, y)$.

The $N^m{}_n(x, y)$ thus appeared to read (I.2.15) can naturally be interpreted as the *coefficients of the non-linear connection produced by the angle in the space \mathcal{F}^N studied on the general indicatrix-inhomogeneous level.*

Because of the conformal flatness of the tangent Riemannian spaces $\mathcal{R}_{\{x\}}$, the Finsler space \mathcal{F}^N involves the associated Riemannian space \mathcal{R}^N and, therefore, the Riemannian connection coefficients $L^m_{ij} = a^m_{ij} + S^m_{ij}$ (shown in (I.1.14)) in which the entered Christoffel symbols a^m_{ij} are to be constructed from the Riemannian metric tensor $a_{mn}(x)$ of the space \mathcal{R}^N ; the notation S^m_{ij} is the torsion tensor.

With the knowledge of the coefficients $N^m_n(x, y)$, we can straightforwardly evaluate the derivative coefficients N^k_{im} and express the Finslerian connection coefficients T^k_{im} through the Riemannian connection coefficients $L^m_{ij} = L^m_{ij}(x)$ and the function $H = H(x)$ (by the help of the formulas (I.1.33) and (I.2.18)).

The coefficients T^k_{im} involve the deflection tensor $\Delta^k_{im} = -N^k_{im} - T^k_{im}$ which is non-vanishing as far as $H_i \neq 0$, namely $\Delta^k_{im} = (1/H)H_i h^k_m$. There arises the covariant derivative \mathcal{T} , which properties are listed in (I.1.37)-(I.1.40). In distinction from the connection developed in the indicatrix-homogeneous case, the \mathcal{T} -connection obtained is no more deflectionless. Nevertheless, the \mathcal{T} -connection is metrical and the equality $N^m_j = -T^m_{ji}y^i$ holds.

In this way, the metrical non-linear Finsler connection $\mathcal{F}N = \{N^m_i, T^m_{ij}\}$ is induced in the space \mathcal{F}^N from the metrical linear connection $\mathcal{R}L = \{L^m_j, L^m_{ij}\}$ evidenced in the Riemannian space \mathcal{R}^N , where $L^m_j = -L^m_{ji}y^i$. The involved function $H = H(x)$ may be an arbitrary smooth function of x .

The Finsler connection $\mathcal{F}N = \{N^m_i, T^m_{ij}\}$ can be understood to be a result of an appropriate nonlinear deformation of the connection $\mathcal{R}L = \{L^m_j, L^m_{ij}\}$. It is the transformation $y = \mathbf{C}(x, \bar{y})$ that represents the deformation said.

In other words, in the Finsler space \mathcal{F}^N we evidence the phenomenon that the metrical non-linear angle-preserving connection is the \mathbf{C} -deformation of the metrical linear connection applicable in the Riemannian space \mathcal{R}^N :

$$\mathcal{F}N = \mathbf{C} \cdot \mathcal{R}L.$$

We shall show that the \mathbf{C} -deformation is \mathcal{T} -covariant constant: $\mathcal{T} \cdot \mathbf{C} = 0$. Also, the covariant derivative \mathcal{T} is the manifestation of the *transitivity* of the connection under this transformation, in short, $\mathcal{T} = \mathcal{C} \cdot \nabla$, where ∇ is the covariant derivative applicable in the Riemannian space \mathcal{R}^N .

In the Riemannian geometry we have merely $H = 1$. In the Finsler space \mathcal{F}^N , the scalar $H(x)$ plays the role of the parameter which changes the indicatrix curvature value. Varying the scalar $H(x)$ evokes the changes in the Finsler space \mathcal{F}^N .

In the theory of Finsler spaces the notion of connection was studied on the basis of various convenient sets of axioms (see [1-5] and references therein). Regarding the significance of the angle notion, the important step was made in [6] where in processes of studying implications of the two-vector angle defined by area, the theorem was proved which states that a diffeomorphism between two Finsler spaces is an isometry iff it keeps the angle. This Tamásy's theorem clearly substantiates the idea to develop the Finsler connection from the Finsler two-vector angle, possibly on the analogy of the Riemannian geometry.

To meet new methods of applications, the interesting chain of linear connections was introduced and studied in [3]. It was emphasized that in the Riemannian geometry we have naturally the metrical and linear connection applicable on the tangent bundle of the variables x, y . Like to the constructions developed in the preceding work [10,11] dealt

with the indicatrix-homogeneous case, in the present indicatrix-inhomogeneous study of the space \mathcal{F}^N the export of this connection generates the required Finsler connection.

By performing the comparison between the commutators of the obtained Finsler covariant derivative \mathcal{T} and the commutators of the underlined Riemannian covariant derivative ∇ , *not* assuming $H = \text{const}$ so that $H(x)$ is permitted to be an arbitrary smooth function of x , the associated curvature tensor $\rho_k^n{}_{ij}$ can straightforwardly be derived.

The Finsleroid case of the space \mathcal{F}^N provides us with the example when the key transformation $y = \mathbf{C}(x, \bar{y})$ is known explicitly. Therefore, we can straightforwardly apply the developed indicatrix-inhomogeneous theory taking the metric function of the Finsleroid type. The explicit representations for the respective Finsleroid coefficients $N^m{}_n$, as well as for the entailed derivative coefficients $N^k{}_{im}$ and $N^k{}_{imn}$, are found. Thus we have got prepared the connection $\mathcal{F}N$ in the Finsleroid space at our disposal with an arbitrary input scalar $H(x)$.

Below we are interested in spaces of the dimension $N \geq 3$. The two-dimensional case has been studied in [8,9].

Chapter I. The Outline

I.1. Basic representations

For a given function Finsler metric function $F = F(x, y)$ we can construct the covariant tangent vector $\hat{y} = \{y_i\}$ and the Finslerian metric tensor $\{g_{ij}\}$ in the conventional way: $y_i := (1/2)\partial F^2/\partial y^i$ and $g_{ij} := \partial y_i/\partial y^j$. The contravariant tensor $\{g^{ij}\}$ is defined by the reciprocity conditions $g_{ij}g^{jk} = \delta_i^k$, where δ stands for the Kronecker symbol. The indices i, j, \dots refer to local admissible coordinates $\{x^i\}$ on the base manifold M . We shall also use the tensor $C_{ijk} = (1/2)\partial g_{ij}/\partial y^k$. By l we shall denote the unit vectors, namely, $l = y/F(x, y)$, such that $F(x, l) = 1$.

Let U_x be a simply connected and geodesically complete region on the indicatrix \mathcal{I}_x supported by a point $x \in M$. Any point pair $u_1, u_2 \in U_x$ can be joined by the respective arc $\mathcal{A}_{\{x\}}(l_1, l_2) \subset \mathcal{I}_x$ of the Riemannian geodesic line drawn on U_x . By identifying the length of the arc with the angle notion we arrive at the **geodesic-arc angle** $\alpha_{\{x\}}(y_1, y_2)$, where $y_1, y_2 \in T_x M$ are two vectors issuing from the origin $0 \in T_x M$ and possessing the property that their direction rays $0y_1$ and $0y_2$ intersect the indicatrix at the point pair $u_1, u_2 \in U_x$. We obtain

$$\alpha_{\{x\}}(y_1, y_2) = \|\mathcal{A}_{\{x\}}(l_1, l_2)\|. \quad (\text{I.1.1})$$

The coefficients $N^k{}_i = N^k{}_i(x, y)$ are required to construct the operator

$$d_i := \frac{\partial}{\partial x^i} + N^k{}_i \frac{\partial}{\partial y^k}. \quad (\text{I.1.2})$$

These coefficients are assumed naturally to be positively homogeneous of degree 1 with respect to the vector argument y .

The derivative coefficients

$$N^k{}_{nm} = \frac{\partial N^k{}_n}{\partial y^m}, \quad N^k{}_{nmj} = \frac{\partial N^k{}_{nm}}{\partial y^j} \quad (\text{I.1.3})$$

possess the identities

$$N^k{}_{nm}y^m = N^k{}_n, \quad N^k{}_{nmj}y^m = N^k{}_{nmj}y^j = 0, \quad N^k{}_{nmj} = N^k{}_{njm}. \quad (\text{I.1.4})$$

The coefficients are used to construct the covariant derivatives

$$\mathcal{D}_k F := d_k F, \quad \mathcal{D}_k l^m := d_k l^m - N^m{}_{kn} l^n, \quad \mathcal{D}_k l_m := d_k l_m + N^h{}_{km} l_h, \quad (\text{I.1.5})$$

and

$$\mathcal{D}_k g_{mn} := d_k g_{mn} + N^h{}_{km} g_{hn} + N^h{}_{kn} g_{mh}. \quad (\text{I.1.6})$$

The identities

$$\frac{\partial \mathcal{D}_k F}{\partial y^m} = \mathcal{D}_k l_m, \quad \frac{\partial \mathcal{D}_k l_m}{\partial y^n} = \mathcal{D}_k g_{mn} + l_h N^h{}_{kmn} \quad (\text{I.1.7})$$

are obviously valid, together with

$$\frac{\partial \mathcal{D}_i g_{mn}}{\partial y^j} = 2\mathcal{D}_i C_{mnj} + N^t{}_{imj} g_{tn} + N^t{}_{inj} g_{mt}, \quad (\text{I.1.8})$$

where

$$\mathcal{D}_i C_{mnj} := d_i C_{mnj} + N^t{}_{ij} C_{mnt} + N^t{}_{im} C_{tnj} + N^t{}_{in} C_{mtj}. \quad (\text{I.1.9})$$

In addition to the Finsler metric tensor g_{mn} , we shall use also the tensor

$$h_{mn} = g_{mn} - l_m l_n \quad (\text{I.1.10})$$

which possesses the property $h_{mn}y^n = 0$. The covariant derivative of this tensor will be constructed in the manner similar to (1.6), namely

$$\mathcal{D}_k h_{mn} := d_k h_{mn} + N^h{}_{km} h_{hn} + N^h{}_{kn} h_{mh}.$$

To deal with the two-vector angle $\alpha = \alpha_{\{x\}}(y_1, y_2)$, we merely extend the operator d_i in the *separable way*, namely

$$d_i = \frac{\partial}{\partial x^i} + N^k{}_i(x, y_1) \frac{\partial}{\partial y_1^k} + N^k{}_i(x, y_2) \frac{\partial}{\partial y_2^k}, \quad y_1, y_2 \in T_x M, \quad (\text{I.1.11})$$

and introduce the covariant derivative $\mathcal{D}_i \alpha$ according to

$$\mathcal{D}_i \alpha = d_i \alpha. \quad (\text{I.1.12})$$

In the Riemannian geometry we have the separable operator

$$d_i^{\text{Riem}} = \frac{\partial}{\partial x^i} + L^k{}_i(x, y_1) \frac{\partial}{\partial y_1^k} + L^k{}_i(x, y_2) \frac{\partial}{\partial y_2^k}, \quad y_1, y_2 \in T_x M, \quad (\text{I.1.13})$$

with the linear coefficients $L^k{}_i(x, y_1) = -L^k{}_{ij}(x)y_1^j$ and $L^k{}_i(x, y_2) = -L^k{}_{ij}(x)y_2^j$ obtained from the Riemannian connection coefficients

$$L^m{}_{ij} = a^m{}_{ij} + S^m{}_{ij}, \quad (\text{I.1.14})$$

where $a^m_{ij} = a^m_{ij}(x)$ stands for the Christoffel symbols constructed from the Riemannian metric tensor $a_{mn}(x)$ and $S^m_{ij} = S^m_{ij}(x)$ is an arbitrary *torsion tensor*: $S_{mij} = -S_{jim}$ with $S_{mij} = a_{mh}S^h_{ij}$. When applied to the Riemannian two-vector angle $\alpha^{\text{Riem}}_{\{x\}}(y_1, y_2) = a_{mn}(x)y_1^m y_2^n / S_1 S_2$, where $S_1 = \sqrt{a_{mn}(x)y_1^m y_1^n}$ and $S_2 = \sqrt{a_{mn}(x)y_2^m y_2^n}$, the operator reveals the fundamental vanishing property

$$d_i^{\text{Riem}} \alpha^{\text{Riem}}_{\{x\}}(y_1, y_2) = 0, \quad y_1, y_2 \in T_x M.$$

By analogy, one may assume that the Finsler coefficients N^k_i fulfill the *separable angle-preservation equation*

$$\mathcal{D}_i \alpha = 0 \tag{I.1.15}$$

to try developing the theory in which the properties

$$\mathcal{D}_k F = 0, \quad \mathcal{D}_k l_m = 0, \quad \mathcal{D}_k l^m = 0, \tag{I.1.16}$$

together with the metricity

$$\mathcal{D}_k g_{mn} = 0 \tag{I.1.17}$$

hold fine. This metricity, taken in conjunction with the identities indicated in (1.7), just entails the vanishing

$$l_h N^h_{kmn} = 0. \tag{I.1.18}$$

The following valuable implication can be deduced from angle by applying the coincidence-limit method exposed in Section 3.2 in [12]: *In any Finsler space the vanishing assumption $d_i \alpha = 0$ of the separable type entails the equality*

$$\mathcal{D}_i h_{mn} = \frac{2}{F} h_{mn} d_i F \tag{I.1.19}$$

(take below the formula (1.29), keeping $H = \text{const}$). If we additionally postulate $d_i F = 0$, we obtain $\mathcal{D}_i h_{mn} = 0$ and, therefore, $\mathcal{D}_i g_{mn} = 0$.

Thus, starting with the separable angle-preservation equation leads to the following implication:

$$\text{PRESERVATION OF ANGLE AND LENGTH} \implies \text{METRICITY}, \tag{I.1.20}$$

that is, the two conditions $d_i \alpha = 0$ and $d_i F = 0$ entail $\mathcal{D}_i g_{mn} = 0$.

When $\mathcal{D}_i g_{mn} = 0$, the identity (1.8) communicates the validity of the vanishing

$$2\mathcal{D}_i C_{mnj} + N^t_{imj} g_{tn} + N^t_{inj} g_{mt} = 0, \tag{I.1.21}$$

which in turn entails that, because the tensor C_{mnj} is totally symmetric, the tensor

$$N_{nimj} := N^t_{imj} g_{tn}$$

must be totally symmetric with respect to the subscripts n, m, j :

$$N_{nimj} = N_{minj} = N_{jimn} = N_{nijm}, \quad (\text{I.1.22})$$

and whence

$$N^k_{imn} = -\mathcal{D}_i C^k_{mn}, \quad (\text{I.1.23})$$

where

$$\mathcal{D}_i C^k_{mn} := d_i C^k_{mn} - N^k_{it} C^t_{mn} + N^t_{im} C^k_{tn} + N^t_{in} C^k_{mt}.$$

With the representation (1.23), the vanishing (1.18) can be regarded as a direct implication of the identity $y^k C_{knj} = 0$ shown by the tensor C_{knj} .

Thus, *in any Finsler space the two conditions $d_i \alpha = 0$ and $d_i F = 0$ entail the representation (1.23) for the coefficients N^k_{imn} .*

By differentiating these coefficients with respect to y^j and making the interchange of the indices m, j , and also noting that $\partial N^k_{imn} / \partial y^j - \partial N^k_{ijn} / \partial y^m = 0$ and

$$\frac{\partial C^k_{mn}}{\partial y^j} - \frac{\partial C^k_{jn}}{\partial y^m} = -2 (C^h_{nm} C^k_{hj} - C^h_{nj} C^k_{hm}),$$

from (1.23) we can arrive at the following vanishing after a short evaluation:

$$\mathcal{D}_i S_n^k{}_{jm} = 0, \quad \text{where } S_n^k{}_{ij} = (C^h_{nj} C^k_{hi} - C^h_{ni} C^k_{hj}) F^2. \quad (\text{I.1.24})$$

However, there are no reasons to trust that the separable form (1.15) for the angle preservation is applicable in general to any Finsler space. For it might happen that the equation (1.15) doesn't permit any solution with respect to the coefficients $N^k_i = N^k_i(x, y)$. Indeed, the formula (1.24) tells us that the following proposition is true.

Proposition I.1.1. *One is entitled to hope to determine the coefficients N^m_n of a Finsler space from the separable equation $d_i \alpha = 0$ supplemented by the condition $d_i F = 0$ if only the Finsler space possesses the property $\mathcal{D}_i S_n^k{}_{jm} = 0$.*

Clearly, the vanishing $\mathcal{D}_i S_n^k{}_{jm} = 0$ can be realized in but rare particular cases of the Finsler space.

In this connection it can be of help to introduce a *characteristic indicatrix scale* $R(x)$ in each tangent space to normalize the angle. If the volume $V_{\mathcal{I}_x}$ of the Finslerian indicatrix $\mathcal{I}_x \subset T_x M$ is finite, it is attractive to obtain the scale by the help of the equality

$$V_{\mathcal{I}_x} = C_1 (R(x))^{N-1}, \quad C_1 = \text{const}. \quad (\text{I.1.25})$$

In this case the $R(x)$ has the geometrical meaning of the *radius of the indicatrix* supported by p. x .

In this respect, there is the deep qualitative distinction of the Finsler geometry from the Riemannian geometry. Namely, in the latter geometry we have simply $V_{\mathcal{I}_x} = \text{const}$, and whence $R = \text{const}$. The new reality that the value of $V_{\mathcal{I}_x}$ may vary from point to point of the background manifold M arises in the Finsler geometry, in which case the R may be a function of x .

The $R(x)$ thus appeared proposes naturally the scale factor in the tangent Riemannian space $\mathcal{R}_{\{x\}}$ supported by the point x .

This motivation suggests the idea to replace the above angle $\alpha_{\{x\}}(y_1, y_2)$ by the *normalized angle*

$$\alpha_{\{x\}}^{\{H(x)\}}(y_1, y_2) := H(x)\alpha_{\{x\}}(y_1, y_2), \quad y_1, y_2 \in T_x M, \quad (\text{I.1.26})$$

where we have introduced the scalar $H(x) = 1/R(x)$, to use the preservation equation

$$d_i \alpha_{\{x\}}^{\{H(x)\}}(y_1, y_2) = 0 \quad (\text{I.1.27})$$

instead of $d_i \alpha_{\{x\}}(y_1, y_2) = 0$ formulated in (1.15). The preservation law (1.27) can be written in the *recurrent* form

$$d_i \alpha + \frac{1}{H} H_i \alpha = 0. \quad (\text{I.1.28})$$

The d_i is the operator (1.11) and $H_i = \partial H / \partial x^i$.

Since the angle $\alpha_{\{x\}}(y_1, y_2)$ is measured by the indicatrix arc length, it seems quite natural to normalize the angle by means of the characteristic scale factor, according to (1.26).

To elucidate patterns, it proves being of great help to apply the coincidence-limit method (see Section 3.2 in [12]). Namely, with the function $E = (1/2)\alpha^2$ the recurrent preservation $d_i \alpha + (1/H)H_i \alpha = 0$ proposed by (1.28) entails the following E -equation

$$\frac{\partial E}{\partial x^i} + N^k_{1i} \frac{\partial E}{\partial y_1^k} + N^k_{2i} \frac{\partial E}{\partial y_2^k} = -\frac{2}{H} H_i E,$$

where $N^k_{1i} = N^k_i(x, y_1)$ and $N^k_{2i} = N^k_i(x, y_2)$. Evaluating various partial derivatives of this E -equation with respect to y_1 and y_2 and finding the coincidence limits when $y_2 \rightarrow y_1$, we can obtain a valuable information of the tensors of the Finsler space. Performing the required evaluations on the level of the second-order partial derivatives $\partial^2 / \partial y_1^m \partial y_2^n$, and, then, applying the operation $y_2 \rightarrow y_1$ to the resultant expressions, it is possible to arrive at the general conclusion that *in any Finsler space the vanishing assumption $d_i \alpha + (1/H)H_i \alpha = 0$ entails the equality*

$$\mathcal{D}_i h_{mn} = \frac{2}{F} h_{mn} d_i F - \frac{2}{H} H_i h_{mn}. \quad (\text{I.1.29})$$

The formula (1.29) has been derived in Appendix E in all detail by performing required long substitutions (see (E.37) in Appendix E).

By differentiating the equality (1.29) with respect to y^j , it is possible to obtain the coefficients N^k_{imn} . In this way, when the vanishing $d_i F = 0$ is also keeping valid, simple direct evaluations yield the representation

$$N^k_{imn} = \frac{2}{H} H_i \frac{1}{F} l^k h_{mn} - \mathcal{D}_i C^k_{mn}, \quad (\text{I.1.30})$$

which extends the previous (1.23). The symmetry (1.22) is now replaced by

$$N^t_{imj}g_{tn} - \frac{2}{H}H_i\frac{1}{F}h_{mj}l_n = N^t_{imn}g_{tj} - \frac{2}{H}H_i\frac{1}{F}h_{mn}l_j.$$

Instead of the vanishing (1.18) we obtain

$$FN^k_{im}l_k = \frac{2}{H}H_ih_{mn}. \quad (\text{I.1.31})$$

The vanishing $\mathcal{D}_i S_n^k{}_{jm} = 0$ indicated in (1.24) is now extended, namely the above representation (1.30) straightforwardly entails the equality

$$\mathcal{D}_i S_n^k{}_{jm} = -\frac{2}{H}H_i(h_j^k h_{mn} - h_m^k h_{jn}).$$

From (1.29) we can conclude that when $d_i F = 0$ we have

$$\mathcal{D}_i g_{mn} = -\frac{2}{H}H_i h_{mn} \quad (\text{I.1.32})$$

at an arbitrary smooth function $H = H(x)$.

The equality (1.32) suggests us to introduce the *total connection coefficients*

$$T^k{}_{im} = -N^k{}_{im} - \frac{1}{H}H_i h_m^k, \quad (\text{I.1.33})$$

so that the *deflection tensor*

$$\Delta^k{}_{im} \stackrel{\text{def}}{=} -N^k{}_{im} - T^k{}_{im} \quad (\text{I.1.34})$$

is non-vanishing as far as $H_i \neq 0$, namely

$$\Delta^k{}_{im} = \frac{1}{H}H_i h_m^k. \quad (\text{I.1.35})$$

It follows that

$$T^k{}_{im}y^m = -N^k{}_{im}y^m \equiv -N^k{}_i, \quad l_k T^k{}_{im} = -l_k N^k{}_{im}. \quad (\text{I.1.36})$$

There arises the *total covariant derivative* \mathcal{T}_i , showing the properties

$$\mathcal{T}_i F = 0, \quad \mathcal{T}_i l_m = 0, \quad \mathcal{T}_i l^m = 0, \quad (\text{I.1.37})$$

and the *metricity*

$$\mathcal{T}_i g_{nm} = 0, \quad (\text{I.1.38})$$

where

$$\mathcal{T}_i F \stackrel{\text{def}}{=} d_i F, \quad \mathcal{T}_i l_m \stackrel{\text{def}}{=} d_i l_m - T^h{}_{im} l_h, \quad \mathcal{T}_i l^m \stackrel{\text{def}}{=} d_i l^m + T^m{}_{ih} l^h, \quad (\text{I.1.39})$$

and

$$\mathcal{T}_i g_{nm} \stackrel{\text{def}}{=} d_i g_{nm} - T^h_{im} g_{hn} - T^h_{in} g_{hm}. \quad (\text{I.1.40})$$

In all the previous formulas started with (1.26), the $H(x)$ was an arbitrary smooth scalar not related anyhow to the indicatrix curvature, the constancy of the indicatrix curvature was not implied, and the Finsler space was arbitrary.

If the indicatrix of a Finsler space is a space of constant curvature at any point $x \in M$, we say that the Finsler space is the \mathcal{F}^N -space, where $N \geq 3$ is the dimension of the space.

The interest to the Finsler space \mathcal{F}^N is motivated by the following important observations. Given an arbitrary Finsler space of any dimension $N \geq 3$. The tangent Riemannian space $\mathcal{R}_{\{x\}} \subset T_x M$ is conformally flat if and only if the indicatrix $\mathcal{I}_x \subset T_x M$ is a space of constant curvature, assuming naturally that the involved conformal multiplier is homogeneous with respect to the argument y . The dependence of the conformal multiplier on the variable y is presented by the power of the Finsler metric function. The remarkable equality $\mathcal{C}_{\text{Ind.}} \equiv H^2$ ensues. These observations form the content of Proposition II.2.1 (formulated and proved in Section II.2 of Chapter II), which extends Proposition 2.1 of the preceding work [10,11] in the following essential aspect.

In [10,11], the assumption was made that the respective conformal multiplier is of the power dependence on the Finsler metric function, in accordance with the representations indicated in the formula (II.2.3) of Section II.2. In proving Proposition II.2.1 in Section II.2, we outline the reasoning line which explains that the representations are actually the direct consequences of the property that the indicatrices are spaces of constant curvature.

We say that the Finsler space \mathcal{F}^N is *indicatrix-homogeneous* if $\mathcal{C}_{\text{Ind.}} = \text{const.}$ In this case, the deflectionless connection has been derived from the separable angle-preservation equation in the preceding work [10,11].

Alternatively, the Finsler space \mathcal{F}^N is said to be *indicatrix-inhomogeneous* if $\mathcal{C}_{\text{Ind.}} = \mathcal{C}_{\text{Ind.}}(x)$. On this level, because of the equality $\mathcal{C}_{\text{Ind.}} \equiv H^2$, we have $H = H(x)$ and $H_i \neq 0$.

On the indicatrix-inhomogeneous level of study of the Finsler space \mathcal{F}^N with $d_i F = 0$ the separable preservation law for the angle is *impossible to introduce*. Indeed, the law entails the metricity $\mathcal{D}_i g_{mn} = 0$ of the deflectionless type (see (1.17) and the definition (1.6)), together with the representation (1.23) for the coefficients N^k_{imn} and the vanishing $\mathcal{D}_i S_n^k{}_{jm} = 0$, where $S_n^k{}_{ij} = (C^h_{nj} C^k_{hi} - C^h_{ni} C^k_{hj}) F^2$ (see (1.24)). It is known that the indicatrix is a space of constant curvature if and only if the last tensor fulfills the equality $S_n^k{}_{ij} = C(h_{nj} h_i^k - h_{ni} h_j^k)$ with the factor C which is independent of y , in which case $\mathcal{C}_{\text{Ind.}} = 1 - C$ (see Section 5.8 in [1]). In the Finsler space \mathcal{F}^N , we have $\mathcal{C}_{\text{Ind.}} = H^2$. The two vanishings $\mathcal{D}_i g_{mn} = 0$ and $\mathcal{D}_i F = 0$ entail $\mathcal{D}_i h_{mn} = 0$. Whence from $\mathcal{D}_i S_n^k{}_{jm} = 0$ it follows that $H_i = 0$.

If, however, we start with recurrent preservation law supplemented by the vanishing condition $\mathcal{D}_i F = 0$, then from (1.29) we have $\mathcal{D}_i h_{mn} = -(2/H) H_i h_{mn}$. Applying the covariant derivative \mathcal{D}_i to the tensor $S_n^k{}_{ij} = C(h_{nj} h_i^k - h_{ni} h_j^k)$ and taking into account that $C = 1 - H^2$, after short evaluations we now arrive at the equality

$$\mathcal{D}_i S_n^k{}_{jm} = -\frac{2}{H} H_i (h_j^k h_{mn} - h_m^k h_{jn})$$

which is equivalent to the implication written below (1.31). Thus, the following proposition is valid.

Proposition I.1.2. *The recurrent-type preservation (1.28) of the angle, that is, $d_i\alpha + (1/H)H_i\alpha = 0$, is reconciled with the indicatrix-inhomogeneous Finsler space \mathcal{F}^N at any scalar $H = H(x)$ obtainable from the identification $\mathcal{C}_{\text{Ind.}} = H^2$.*

The observations motivate us to go to the preservation law (1.27) which is not separable from the standpoint of the indicatrix-arc angle $\alpha_{\{x\}}(y_1, y_2)$, whenever $H \neq \text{const.}$

In so doing, the coefficients N^m_n of the Finsler space \mathcal{F}^N are obtained to read (I.2.16) in Section I.2. They don't involve explicitly the gradients H_n . If, however, we expand the partial derivatives $\partial/\partial x^n$ which enter the right-hand part of (I.2.16), the coefficients will break down into two parts:

$$N^m_n = N^{Im}_n + \check{N}^m_n, \quad \check{N}^m_n = \check{N}^m H_n. \quad (\text{I.1.41})$$

Here, the first part N^{Im}_n are the coefficients of the indicatrix-homogeneous case (given by the formula (2.30) in [10], and by the formula (2.36) in [11]) in which the constant H has been merely replaced by arbitrary $H(x)$, and the vector field \check{N}^m does not involve any gradient of $H(x)$. We may say that the coefficients N^m_n are of the *linear dependence* on the gradient H_n .

The entailed coefficients N^k_{mn} are given by the representation (II.3.32) of Chapter II which is applicable to any indicatrix-inhomogeneous Finsler space \mathcal{F}^N . It is also possible to evaluate explicitly the derivative coefficients $N^k_{mni} = \partial N^k_{mn}/\partial y^i$. The required evaluations lead straightforwardly to the validity of the representation (1.30) in the \mathcal{F}^N -space with an arbitrary smooth function $H(x)$, provided the vanishing $d_n F = 0$ is assumed (see Proposition II.3.5 in Chapter II).

Having evaluated the coefficients N^k_{mn} , we obtain from (1.33) the total connection coefficients T^k_{im} thereby solving the problem of finding the connection in the \mathcal{F}^N -space at the indicatrix-inhomogeneous level. The coefficients T^k_{im} involve the deflection tensor Δ^k_{im} indicated in (1.34) and (1.35). There arises the covariant derivative \mathcal{T} , which properties are listed in (1.36)-(1.40).

Section I.2 gives a brief summary of Chapter II.

The formula (I.2.16) indicates the representation of the coefficients N^m_n which is valid for an arbitrary Finsler space of the type \mathcal{F}^N . The representation involves the vector field U^i which realizes the key transformation $y = \mathbf{C}(x, \bar{y})$ indicated in (I.2.1). Given a particular Finsler space of the type \mathcal{F}^N , the formula (I.2.16) yields the coefficients N^m_n in a completely explicit way when the respective field U^i is known.

The Finsleroid case to which Section I.3 is devoted provides us with such an example, for the required field U^i is explicitly given, namely by means of the representation (I.3.20) (which was earlier found in Section 6 of [7]). Therefore, we can straightforwardly apply the developed theory of the \mathcal{F}^N -space to the metric function of the Finsleroid type. The expansion (1.41) for the respective Finsleroid coefficients N^m_n has been evaluated. The explicit representation of the entailed derivative coefficients N^k_{im} is indicated. The respective validity of the representations (1.29) and (1.30) of the tensors $\mathcal{D}_i h_{mn}$ and N^k_{imn} on the indicatrix-inhomogeneous level of study of the Finsleroid space has been verified by direct evaluations presented in detail.

Several Appendices are added in which numerous fragments of the underlined evaluations have been displayed.

I.2. Indicatrix of constant curvature

Let M be the base manifold, such that $\mathcal{F}^N = (M, F)$, where $F = F(x, y)$ is the Finsler metric function and $N \geq 3$ is the dimension of the space. If the indicatrix of a Finsler space is a space of constant curvature, we say that the Finsler space is the \mathcal{F}^N -space. Denote by $\mathcal{C}_{\text{Ind.}}$ the value of curvature of the indicatrix supported by the point $x \in M$. If $\mathcal{C}_{\text{Ind.}}$ is a constant over the manifold M , we say that the space \mathcal{F}^N is of the *indicatrix-homogeneous* case.

In general, the value $\mathcal{C}_{\text{Ind.}}$ may vary from point to point of M , in which case we say that the space \mathcal{F}^N is of the *indicatrix-inhomogeneous* type. The possibility is characterized by a function $\mathcal{C}_{\text{Ind.}} = \mathcal{C}_{\text{Ind.}}(x)$ such that the derivative $\partial \mathcal{C}_{\text{Ind.}} / \partial x^i$ does not vanish identically.

In such spaces, the transformation

$$y = \mathbf{C}(x, \bar{y}), \quad y, \bar{y} \in T_x M, \quad (\text{I.2.1})$$

can be proposed which maps the tangent vectors $y \in T_x M$ into the tangent vectors of the same tangent space $T_x M$, subject to the following conditions. The transformation is non-linear with respect to \bar{y} . Non-singularity and sufficient smoothness are implied. Also, the transformation is positively homogeneous of a degree $H(x)$ regarding dependence on tangent vectors y . Each tangent Riemannian space $\mathcal{R}_{\{x\}} = \{T_x M, g_{\{x\}}(y)\}$ is conformally transformed to Euclidean space, to be denoted by $\mathcal{E}_{\{x\}}$. The distribution of the last spaces $\mathcal{E}_{\{x\}}$ over the base manifold M composes the *associated Riemannian space*, which we denote by $\mathcal{R}^N = (M, S)$, where $S = \sqrt{a_{mn}(x)y^m y^n}$ is the Riemannian metric constructed from the metric tensor $a_{mn}(x)$ of the space $\mathcal{E}_{\{x\}}$.

Under these conditions, the scalar $H(x)$ can be taken from the identification

$$\mathcal{C}_{\text{Ind.}} \equiv H^2. \quad (\text{I.2.2})$$

The equality

$$S(x, \bar{y}) = (F(x, y))^{H(x)} \quad (\text{I.2.3})$$

arises (see (II.2.10)), which validates the indicatrix correspondence to the Euclidean sphere; $S(x, \bar{y}) = \sqrt{a_{mn}(x)\bar{y}^m \bar{y}^n}$. The relevant conformal multiplier p^2 is constructed from the Finsler metric function F , according to

$$p = \frac{1}{H} F^{1-H}. \quad (\text{I.2.4})$$

We take $1 > H > 0$ for definiteness, the extension of the approach to other values of H being a straightforward task.

If $f(x, y)$ is the involved conformal multiplier in the tangent Riemannian space $\mathcal{R}_{\{x\}}$, then the equality

$$g_{\{x\}}(y) = f(x, y) u_{\{x\}}(y)$$

should introduce the tensor $u_{\{x\}}(y)$ which associated Riemannian curvature tensor vanishes identically. The function $f(x, y)$ is assumed naturally to be homogeneous with respect to the argument y . Denoting the homogeneity degree of $f(x, y)$ by means of $2a(x)$, we just conclude that the difference $1 - a$ is exactly the homogeneity degree of the transformation (2.1) considered, that is,

$$H = 1 - a.$$

The following assertions are valid. A Finsler space is the \mathcal{F}^N -space if and only if the indicatrix of the Finsler space is a space of constant curvature. The dependence of the multiplier f on the variable y is presented by the power of the Finsler metric function F (see Proposition II.2.1 in Section 2 of Chapter II).

The respective two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ proves to be obtainable from the angle $\alpha_{\{x\}}^{\text{Riem}}(y_1, y_2)$ operative in the Riemannian space, namely the simple equality

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)} \alpha_{\{x\}}^{\text{Riem}}(\bar{y}_1, \bar{y}_2) \quad (\text{I.2.5})$$

(see (II.2.51)-(II.2.52)) is valid.

We locally represent the transformation (2.1) by means of the functions

$$y^i = y^i(x, t), \quad t^n \equiv \bar{y}^n. \quad (\text{I.2.6})$$

The homogeneity entails $y^i(x, kt) = k^{1/H} y^i(x, t)$ with $k > 0$ and $\forall t$, together with $y_n^i t^n = (1/H) y^i$, where $y_n^i = \partial y^i / \partial t^n$.

The definition

$$U^i \stackrel{\text{def}}{=} (1/S) \bar{y}^i \equiv (1/F^H) \bar{y}^i \quad (\text{I.2.7})$$

introduces the normalized vector, which is obviously unit: $U_i U^i = 1$ and $U_i = a_{ij} U^j$. The zero-degree homogeneity $U^i(x, ky) = U^i(x, y)$ with $k > 0$ and $\forall t$ holds, entailing the identity $U_n^i y^n = 0$ with

$$U_n^i := \frac{\partial U^i}{\partial y^n} = \frac{1}{F^H} t_n^i - \frac{1}{F} H U^i l_n, \quad (\text{I.2.8})$$

where $t_n^i = \partial t^i / \partial y^n$. It follows that

$$F^H U_s^h y_h^k = h_s^k, \quad F^H U_k^i y_t^k = \delta_t^i - U^i U_t, \quad U_i U_n^i = 0. \quad (\text{I.2.9})$$

The vanishing

$$U_i \left(\frac{\partial U^i}{\partial x^n} + L^i_{kn} U^k \right) = 0 \quad (\text{I.2.10})$$

holds obviously, where L^i_{nk} are the Riemannian connection coefficients (I.1.14).

The representation (2.5) of the angle takes on the simple form

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)} \arccos \lambda, \quad \text{with } \lambda = a_{mn}(x) U_1^m U_2^n, \quad (\text{I.2.11})$$

where $U_1^m = U^m(x, y_1)$ and $U_2^m = U^m(x, y_2)$.

When the recurrent preservation $d_i \alpha + (1/H) H_i \alpha = 0$ proposed by (1.28) is applied to the angle given in (2.11), we obtain simply

$$d_i \lambda = 0, \quad (\text{I.2.12})$$

where d_i is the separable operator (1.11). That is, the recurrent preservation law formulated for the Finsler \mathcal{F}^N -space angle $\alpha_{\{x\}}$ given by (2.11) is tantamount to the separable preservation law for the Euclidean angle $\alpha_{\{x\}}^{\text{Riem}} = \arccos \lambda$, whence to the separable preservation law (2.12).

The form of the right-hand part in the formula $\lambda = a_{mn}(x)U_1^m U_2^n$ is such that the law (2.11) is obviously equivalent to the vanishing

$$\mathcal{D}_n U^i = 0 \quad (\text{I.2.13})$$

for the field $U^i = U^i(x, y)$, where we introduced the covariant derivative

$$\mathcal{D}_n U^i := d_n U^i + L^i_{nk} U^k. \quad (\text{I.2.14})$$

Since

$$d_n U^i = \frac{\partial U^i}{\partial x^n} + N^i_{nk} U^k,$$

we arrive at the conclusion that in the \mathcal{F}^N -space, the coefficients N^m_n can unambiguously be found from the equation $d_n (H(x)\alpha_{\{x\}}(y_1, y_2)) = 0$ to be given explicitly by the representation

$$N^m_n = -y_i^m F^H \left(\frac{H}{F} U^i \frac{\partial F}{\partial x^n} + \frac{\partial U^i}{\partial x^n} + (a^i_{nk} + S^i_{nk}) U^k \right) + l^m d_n F \quad (\text{I.2.15})$$

(see (II.3.12) in Chapter II). Here, a^i_{nk} are the Riemannian Christoffel symbols; $S^i_{nk} = S^i_{nk}(x)$ is an arbitrary torsion tensor, that is, the tensor possessing the skew-symmetry property $S_{ink} = -S_{kni}$, where $S_{ink} = a_{ij} S^j_{nk}$.

Whenever $d_n F = 0$, the representation (2.15) takes on the form

$$N^m_n = -l^m \frac{\partial F}{\partial x^n} - y_i^m F^H \left(\frac{\partial U^i}{\partial x^n} + (a^i_{nk} + S^i_{nk}) U^k \right) \quad (\text{I.2.16})$$

(see (II.1.19) in Chapter II). These coefficients N^m_n present the *general solution* to the couple equations $d_n (H(x)\alpha_{\{x\}}(y_1, y_2)) = 0$ and $d_n F = 0$, so that no problem of uniqueness of connection coefficients may be questioned. The entrance of the torsion tensor S^i_{kn} is the only freedom, in complete analogy to the connection coefficients of the Riemannian space.

The evaluations performed in Section II.3 of Chapter II have arrived also at the representation

$$N^m_n = d_n^{\text{Riem}} y^m(x, t) + \frac{1}{H} H_n y^m \ln F \quad (\text{I.2.17})$$

(see (II.3.29) in Chapter II) which is alternative to (2.16); here, $y^m = y^m(x, t)$ are the functions (2.6).

The representations (2.15)-(2.17) involve the gradient H_n and are applicable to any indicatrix-inhomogeneous Finsler space \mathcal{F}^N .

The coefficients N^k_{mn} can be evaluated from (2.16) to read

$$\begin{aligned} N^k_{mn} = & -\frac{1}{F} h_n^k \frac{\partial F}{\partial x^m} - l^k \frac{\partial l_n}{\partial x^m} - C^k_{ns} N^s_m + \frac{1}{F} (l_n h_s^k - (1-H) l^k h_{ns}) N^s_m \\ & - y_h^k F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U_n^s \right) \end{aligned} \quad (\text{I.2.18})$$

(see Proposition II.3.4 in Chapter II). With these coefficients, the validity of the representation (1.30) for the entailed coefficients N^k_{imn} can straightforwardly be verified (see Proposition II.3.5 in Chapter II).

The space \mathcal{F}^N is obtainable from the Riemannian space \mathcal{R}^N by means of the deformation $y = \mathbf{C}(x, \bar{y})$ (see (II.2.1) in Chapter II) which can be presented by the *deformation tensor*

$$C_m^i := p\bar{y}_m^i, \quad (\text{I.2.19})$$

so that

$$g_{mn} = C_m^i C_n^j a_{ij} \quad (\text{I.2.20})$$

and the zero-degree homogeneity

$$C_m^i(x, ky) = C_m^i(x, y), \quad k > 0, \forall y, \quad (\text{I.2.21})$$

holds, together with the identity

$$C_m^i(x, y)y^m = (F(x, y))^{1-H} \bar{y}^i \quad (\text{I.2.22})$$

(see (II.2.24)-(II.2.27)). In Section II.4 we show that the \mathbf{C} -deformation is \mathcal{T} -covariant constant:

$$\mathcal{T} \cdot \mathbf{C} = 0, \quad (\text{I.2.23})$$

where \mathcal{T} designates the covariant derivative introduced by the help of the formulas (I.1.33)-(I.1.40) (see Proposition II.4.1).

Also, the covariant derivative \mathcal{T} is the manifestation of the *transitivity* of the connection under the \mathcal{C} -transformation, in short,

$$\mathcal{T} = \mathcal{C} \cdot \nabla, \quad (\text{I.2.24})$$

where ∇ is the covariant derivative applicable in the background Riemannian space \mathcal{R}^N (see Proposition II.4.2). In other words, in the Finsler space \mathcal{F}^N the metrical non-linear angle-preserving connection is the \mathbf{C} -export of the metrical linear connection (II.1.2) applicable in the space \mathcal{R}^N .

In Section II.5 we perform the attentive comparison between the commutators of the involved Finsler covariant derivative \mathcal{T} and the commutators of the underlined Riemannian covariant derivative ∇ , *not* assuming $H = \text{const}$, such that $H(x)$ can be an arbitrary smooth function of x . In this way, we derive the associated curvature tensor $\rho_k^n{}_{ij}$. Important properties of the tensor are elucidated.

I.3. Reduction to the Finsleroid space

In the Finsleroid case, we make the notation change $H(x) \rightarrow h(x)$.

The scalar $g(x)$ obtained through

$$h(x) = \sqrt{1 - \frac{g^2(x)}{4}}, \quad \text{with} \quad -2 < g(x) < 2, \quad (\text{I.3.1})$$

plays the role of the characteristic parameter.

It follows that

$$g_i = -\frac{4h}{g}h_i, \quad (\text{I.3.2})$$

where $g_i = \partial g / \partial x^i$ and $h_i = \partial h / \partial x^i$.

We assume that in addition to a Riemannian metric $\sqrt{a_{ij}(x)y^iy^j}$ the manifold M admits a non-vanishing 1-form $b = b_i(x)y^i$ of the unit length:

$$a_{ij}(x)b^i(x)b^j(x) = 1, \quad (\text{I.3.3})$$

where $b^i(x) = a^{ij}(x)b_j(x)$. The tensor $a^{ij}(x)$ is reciprocal to $a_{ij}(x)$, so that $a_{ij}a^{jn} = \delta_i^n$, where δ_i^n stands for the Kronecker symbol. We need also the quadratic form

$$B = b^2 + gbq + q^2 \equiv \left(b + \frac{1}{2}gq\right)^2 + h^2q^2, \quad (\text{I.3.4})$$

where

$$q = \sqrt{r_{mn}y^my^n} \quad \text{with} \quad r_{mn} = a_{mn} - b_mb_n, \quad (\text{I.3.5})$$

so that

$$a_{ij}(x)y^iy^j = b^2 + q^2. \quad (\text{I.3.6})$$

We shall also use the scalar

$$\chi = \frac{1}{h} \left(-\arctan \frac{G}{2} + \arctan \frac{L}{hb} \right), \text{ if } b \geq 0; \quad \chi = \frac{1}{h} \left(\pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb} \right), \text{ if } b \leq 0, \quad (\text{I.3.7})$$

with the function $L = q + (g/2)b$ fulfilling the identity

$$L^2 + h^2b^2 = B. \quad (\text{I.3.8})$$

The definition range

$$0 \leq \chi \leq \frac{1}{h}\pi$$

is of value to describe all the tangent space. The normalization in (3.7) is such that

$$\chi|_{y=b} = 0. \quad (\text{I.3.9})$$

The quantity (3.7) can conveniently be written as

$$\chi = \frac{1}{h}f \quad (\text{I.3.10})$$

with the function

$$f = \arccos \frac{A(x, y)}{\sqrt{B(x, y)}} \quad (\text{I.3.11})$$

ranging as follows:

$$0 \leq f \leq \pi. \quad (\text{I.3.12})$$

The Finsleroid-axis vector b^i relates to the value $f = 0$, and the opposed vector $-b^i$ relates to the value $f = \pi$:

$$f = 0 \quad \sim \quad y = b; \quad f = \pi \quad \sim \quad y = -b. \quad (\text{I.3.13})$$

With these ingredients, we construct the Finsler metric function

$$K = \sqrt{B} J, \quad \text{with } J = e^{-\frac{1}{2}g\chi}. \quad (\text{I.3.14})$$

The normalization is such that

$$K(x, b(x)) = 1 \quad (\text{I.3.15})$$

(notice that $q = 0$ at $y^i = b^i$). The positive (not absolute) homogeneity holds: $K(x, \gamma y) = \gamma K(x, y)$ for any $\gamma > 0$ and all admissible (x, y) .

Under these conditions, we call $K(x, y)$ the \mathcal{FF}_g^{PD} -Finsleroid metric function, obtaining the \mathcal{FF}_g^{PD} -Finsler space

$$\mathcal{FF}_g^{PD} := \{M; a_{ij}(x); b_i(x); g(x); K(x, y)\}. \quad (\text{I.3.16})$$

Definition. Within any tangent space $T_x M$, the metric function $K(x, y)$ produces the \mathcal{FF}_g^{PD} -Finsleroid

$$\mathcal{FF}_{g;\{x\}}^{PD} := \{y \in \mathcal{FF}_g^{PD} : y \in T_x M, K(x, y) \leq 1\}. \quad (\text{I.3.17})$$

Definition. The \mathcal{FF}_g^{PD} -Indicatrix $\mathcal{IF}_{g;\{x\}}^{PD} \subset T_x M$ is the boundary of the \mathcal{FF}_g^{PD} -Finsleroid, that is,

$$\mathcal{IF}_{g;\{x\}}^{PD} := \{y \in \mathcal{FF}_g^{PD} : y \in T_x M, K(x, y) = 1\}. \quad (\text{I.3.18})$$

Definition. The scalar $g(x)$ is called the *Finsleroid charge*. The 1-form $b = b_i(x)y^i$ is called the *Finsleroid-axis 1-form*.

The entailed components $y_i := (1/2)\partial K^2/\partial y^i$ of the covariant tangent vector $\hat{y} = \{y_i\}$ can be found in the simple form

$$y_i = (u_i + gqb_i)J^2, \quad (\text{I.3.19})$$

where $u_i = a_{ij}y^j$.

Let us elucidate the structure of the coefficients N_m^k in the Finsleroid case proper. From (6.26) of [7] it follows that the quantity $U^i = (1/K^h)\bar{y}^i$ can explicitly be given by

$$U^i = \left[hv^i + \left(b + \frac{1}{2}gq \right) b^i \right] \frac{1}{\sqrt{B}}, \quad (\text{I.3.20})$$

where $v^i = y^i - bb^i$. So we have

$$\frac{\partial U^i}{\partial g} = -\frac{g}{4h}v^i \frac{1}{\sqrt{B}} + \frac{1}{2}qb^i \frac{1}{\sqrt{B}} - \frac{1}{2B}U^i qb,$$

or

$$\frac{\partial U^i}{\partial g} = -\frac{g}{4h^2}U^i + \frac{g}{4h^2}\left(b + \frac{1}{2}gq\right)b^i\frac{1}{\sqrt{B}} + \frac{1}{2}qb^i\frac{1}{\sqrt{B}} - \frac{1}{2B}U^iqb.$$

Since

$$K^hy_i^mU^i = \frac{1}{h}y^m$$

(a consequence of the homogeneity involved) and

$$K^hy_i^mb^i = \left[b^m + \frac{1}{B}\left(\frac{1}{h}\left(b + \frac{1}{2}gq\right) - b - gq\right)y^m\right]\sqrt{B}$$

(see (D.12) in [7]), we can straightforwardly evaluate the contraction

$$\begin{aligned} K^hy_i^m\frac{\partial U^i}{\partial g} &= -\frac{g}{4h^2}\frac{1}{h}y^m - \frac{1}{2B}\frac{1}{h}qby^m + \frac{g}{4h^2}\left(b + \frac{1}{2}gq\right)b^m + \frac{g}{4h^2}\frac{1}{B}\frac{1}{h}\left(b + \frac{1}{2}gq\right)^2y^m \\ &\quad - \frac{g}{4h^2}\frac{1}{B}\left(b + \frac{1}{2}gq\right)(b + gq)y^m + \frac{1}{2}q\left[b^m + \frac{1}{B}\left(\frac{1}{h}\left(b + \frac{1}{2}gq\right) - b - gq\right)y^m\right]. \end{aligned}$$

Using the equality

$$\left(b + \frac{1}{2}gq\right)^2 = B - h^2q^2$$

(see (3.4)) leads to the representation

$$\begin{aligned} K^hy_i^m\frac{\partial U^i}{\partial g} &= \frac{g}{4h^2}\left(b + \frac{1}{2}gq\right)b^m - \frac{g}{4}\frac{1}{B}\frac{1}{h}q^2y^m - \frac{g}{4h^2}\frac{1}{B}\left(b + \frac{1}{2}gq\right)(b + gq)y^m \\ &\quad + \frac{1}{2}q\left[b^m + \frac{1}{B}\left(\frac{1}{h}\frac{1}{2}gq - b - gq\right)y^m\right], \end{aligned}$$

which can be simplified as follows:

$$\begin{aligned} K^hy_i^m\frac{\partial U^i}{\partial g} &= \frac{g}{4h^2}\left(b + \frac{1}{2}gq\right)b^m - \frac{g}{4h^2}\frac{1}{B}\left(b + \frac{1}{2}gq\right)(b + gq)y^m \\ &\quad + \frac{1}{2h^2}q\left[b^m\left(1 - \frac{g^2}{4}\right) - \frac{1}{B}(b + gq)y^m\left(1 - \frac{g^2}{4}\right)\right] \\ &= \frac{1}{2h^2}\left(q + \frac{1}{2}gb\right)b^m - \frac{1}{2h^2}\frac{1}{B}\left(q + \frac{1}{2}gb\right)(b + gq)y^m, \end{aligned}$$

so that

$$K^hy_i^m\frac{\partial U^i}{\partial g} = \frac{1}{2h^2}\frac{1}{B}\left(q + \frac{1}{2}gb\right)[Bb^m - (b + gq)y^m].$$

By comparing this result with the representation

$$A^m = \frac{N}{2} g \frac{1}{qK} \left[q^2 b^m - (b + gq) v^m \right] \equiv K C^{mn}{}_n$$

(see (A.27) in [7]), we come to

$$K^h y_i^m \frac{\partial U^i}{\partial g} = \frac{1}{h^2} \frac{q}{B} \left(q + \frac{1}{2} g b \right) \frac{K}{N g} A^m. \quad (\text{I.3.21})$$

Therefore, in the Finsleroid case the coefficients $N^k{}_i$ proposed by (I.2.16) are the sum

$$N^k{}_i = N^{\text{Ik}}{}_i + \check{N}^k{}_i, \quad \check{N}^k{}_i = \check{N}^k g_i, \quad (\text{I.3.22})$$

where

$$\check{N}^k = -\frac{1}{h^2} \frac{q}{B} \left(q + \frac{1}{2} g b \right) \frac{K}{N g} A^k - \frac{1}{2} \bar{M} y^k \quad (\text{I.3.23})$$

with \bar{M} coming from

$$\frac{\partial K^2}{\partial g} = \bar{M} K^2. \quad (\text{I.3.24})$$

The torsion tensor $S^k{}_{ij} = S^k{}_{ij}(x)$ has been neglected. The $N^{\text{Ik}}{}_i$ are the coefficients (6.48) of [7] (they can also be found in [10,11]), namely,

$$N^{\text{Ik}}{}_i = \left[\left(b - \frac{1}{h} \left(b + \frac{1}{2} g q \right) \right) \eta^{kj} + \left(\frac{1}{q^2} v^k \left(b - \frac{1}{h} (b + gq) \right) + \left(\frac{1}{h} - 1 \right) b^k \right) y^j \right] \nabla_i b_j - a^k{}_{ij} y^j. \quad (\text{I.3.25})$$

They don't involve the gradient g_i . The tensor

$$\eta^{kn} = a^{kn} - b^k b^n - \frac{1}{q^2} v^k v^n \quad (\text{I.3.26})$$

enters the representation. This tensor obeys the nullification

$$y_k \eta^{kn} = b_k \eta^{kn} = 0. \quad (\text{I.3.27})$$

The designation ∇_i stands for the Riemannian covariant derivative constructed with the help of the Riemannian Christoffel symbols $a^k{}_{ij} = a^k{}_{ij}(x)$.

The $N^{\text{Ik}}{}_i$ are the coefficients $N^k{}_i$ obtained when the condition $h = \text{const}$ which specifies the indicatrix-homogeneous case is postulated.

For the coefficients

$$\check{N}^k{}_{im} = \frac{\partial \check{N}^k{}_i}{\partial y^m}$$

the representation

$$\begin{aligned}\check{N}^k_{im} = & \frac{1}{h^2}g_i\frac{q^2}{2B}\left(1 + \frac{1}{2}g\frac{b}{q} - 2h^2\right)\frac{2}{Ng}A_m l^k + \frac{1}{h^2}g_i\frac{q^2}{2B}\left(1 + \frac{1}{2}g\frac{b}{q}\right)\left(\frac{b}{q} + g\right)h_m^k \\ & + \frac{1}{h^2}g_i\frac{q^2}{2B}\left(\frac{b}{q} + \frac{1}{2}g\right)\frac{2}{Ng}\frac{2}{Ng}A_m A^k - \frac{1}{2}g_i\bar{M}h_m^k + \frac{1}{K}l_m\check{N}^k_i\end{aligned}\quad (\text{I.3.28})$$

is obtained (see Appendix A).

Using (3.28) we find straightforwardly that

$$y_k\frac{\partial^2\check{N}^k_i}{\partial y^m\partial y^n} = \frac{2}{h}h_i h_{mn}. \quad (\text{I.3.29})$$

For the coefficients

$$\check{N}^k_{imn} = \frac{\partial\check{N}^k_{im}}{\partial y^n}$$

the representation

$$\check{N}^k_{imn} = -\frac{g}{2h^2}g_i\frac{1}{K}h_{mn}l^k - \frac{1}{gh^2}g_i\frac{1}{K}A^k_{mn} \quad (\text{I.3.30})$$

can explicitly be derived (see Appendix A); $A^k_{mn} = KC^k_{mn}$.

The full coefficients read

$$N^k_{imn} = \frac{2}{h}h_i\frac{1}{K}l^k h_{mn} - \frac{1}{K}\mathcal{D}_i A^k_{mn} \quad (\text{I.3.31})$$

(see Appendix A). Thus in the Finsleroid case proper we have straightforwardly verified the validity of the representation (1.30).

Chapter II. Phenomenon of indicatrix of constant curvature with $\mathcal{C}_{\text{Ind.}} = \mathcal{C}_{\text{Ind.}}(x)$

II.1. Motivation

In any dimension $N \geq 3$ the Finsler metric function F geometrizes the tangent bundle TM over the base manifold M such that at each point $x \in M$ the tangent space $T_x M$ is endowed with the curvature tensor constructed from the respective Finslerian metric tensor $g_{\{x\}}(y)$ by means of the conventional rule of the Riemannian geometry considering y to be the variable argument. There arises the Riemannian space $\mathcal{R}_{\{x\}} = \{T_x M, g_{\{x\}}(y)\}$ supported by the point $x \in M$ such that $T_x M$ plays the role of the base manifold for the space. We call $\mathcal{R}_{\{x\}}$ the *tangent Riemannian space*.

Given an N -dimensional Riemannian space $\mathcal{R}^N = (M, S)$, where S denotes the Riemannian metric function, one may endeavor to obtain a Finsler space $\mathcal{F}^N = (M, F)$

by applying an appropriate transformation \mathbf{C} to tangent spaces. The base manifold M is keeping the same for both the spaces, \mathcal{R}^N and \mathcal{F}^N .

We assume that the transformation \mathbf{C} is *restrictive*, in the sense that no point $x \in M$ is shifted under the transformation, so that in each tangent space $T_x M$ the deformation maps tangent vectors $y \in T_x M$ into the tangent vectors of the same $T_x M$:

$$y = \mathbf{C}(x, \bar{y}), \quad y, \bar{y} \in T_x M. \quad (\text{II.1.1})$$

In general, this transformation is non-linear with respect to \bar{y} . Non-singularity and sufficient smoothness are always implied.

We may evidence in the Riemannian space \mathcal{R}^N the *metrical linear Riemannian connection* $\mathcal{R}L$, which in terms of local coordinates $\{x^i\}$ introduced in M is given by

$$\mathcal{R}L = \{L^m_j, L^m_{ij}\} : \quad L^m_j = -L^m_{ji}y^i, \quad L^m_{ij} = a^m_{ij} + S^m_{ij}, \quad (\text{II.1.2})$$

where $a^m_{ij} = a^m_{ij}(x)$ stands for the Christoffel symbols constructed from the Riemannian metric tensor $a_{mn}(x)$ of the space \mathcal{R}^N and $S^m_{ij} = S^m_{ij}(x)$ is an arbitrary *torsion tensor*: $S_{mij} = -S_{jim}$ with $S_{mij} = a_{mh}S^h_{ij}$. The respective covariant derivative ∇ can be introduced in the natural way. Namely, considering the (1,1)-type tensor $W^n_m(x, y)$ on the tangent bundle associated to the space \mathcal{R}^N , we can take the definition

$$\nabla_i W^n_m = d_i^{\text{Riem}} W^n_m + L^n_{hi} W^h_m - L^h_{mi} W^n_h, \quad (\text{II.1.3})$$

which involves the action of the operator

$$d_i^{\text{Riem}} = \frac{\partial}{\partial x^i} + L^k_i \frac{\partial}{\partial y^k}. \quad (\text{II.1.4})$$

In the tangent Riemannian space $\mathcal{R}_{\{x\}}$ we can construct from the metric tensor $g_{ij} = g_{ij}(x, y)$ the curvature tensor $\hat{R}_{\{x\}} = \{\hat{R}_n^m{}_{ij}(x, y)\}$ by the help of the ordinary Riemannian method, regarding $\{y^i\}$ as variables. Namely, we obtain the representation

$$\hat{R}_n^m{}_{ij} = \frac{\partial C^m_{ni}}{\partial y^j} - \frac{\partial C^m_{nj}}{\partial y^i} + C^h_{ni} C^m_{hj} - C^h_{nj} C^m_{hi}.$$

Since $\partial C^m_{ni}/\partial y^j - \partial C^m_{nj}/\partial y^i \equiv -2(C^h_{ni} C^m_{hj} - C^h_{nj} C^m_{hi})$, we have simply

$$\hat{R}_n^m{}_{ij} = \frac{1}{F^2} S_n^m{}_{ij},$$

where

$$S_n^m{}_{ij} = (C^h_{nj} C^m_{hi} - C^h_{ni} C^m_{hj}) F^2.$$

The tensor $S_n^m{}_{ij}$ describes the curvature of indicatrix (see Section 5.8 in [1]).

We need the *metrical non-linear Finsler connection* $\mathcal{F}N$, such that

$$\mathcal{F}N = \{N^m_i, T^m_{ij}\} : \quad N^m_i = N^m_i(x, y), \quad T^m_{ij} = T^m_{ij}(x, y), \quad (\text{II.1.5})$$

where the objects $N^m_i(x, y)$ and $T^m_{ij}(x, y)$ are to depend on the variable y in an essentially non-linear way. The adjective “metrical” means that the action of the entailed covariant derivative on the Finsler metric function, and also on the Finsler metric tensor, yields

identically zero. The coefficients N^m_i and T^m_{ij} are assumed to be positively homogeneous regarding the dependence on vectors y , respectively of degree 1 and degree 0.

In the Riemannian limit of the Finsler space, the spaces $\mathcal{R}_{\{x\}}$ are Euclidean spaces and the tensor $g_{\{x\}}(y)$ is independent of y . The conformally flat structure of the spaces $\mathcal{R}_{\{x\}}$ can naturally be taken to treat as the next level of generality of the Finsler space. Can the metrical connection preserving the two-vector angle be introduced on that level?

The deformation of the Riemannian space to the Finsler space proves to be the convenient method of consideration to apply. Namely, when the Riemannian space can be deformed to the Finsler space characterized by the conformally flat structure of the spaces $\mathcal{R}_{\{x\}}$ the positive and clear answer to the above question can be arrived at. The respective conformal multiplier is shown to be a power of the Finsler metric function.

We shall evidence the phenomenon that the used non-linear deformation

$$\mathcal{F}N = \mathbf{C} \cdot \mathcal{R}L \quad (\text{II.1.6})$$

of the Riemannian connection yields the Finsler connection $\mathcal{F}N$ which preserves the Finslerian two-vector angle $\alpha_{\{x\}}(y_1, y_2)$.

II.2. Key observations

Below, *any dimension* $N \geq 3$ is allowable.

Let M be an N -dimensional C^∞ differentiable manifold, $T_x M$ denote the tangent space to M at a point $x \in M$, and $y \in T_x M \setminus 0$ mean tangent vectors. Suppose we are given on the tangent bundle TM a Riemannian metric S . Denote by $\mathcal{R}^N = (M, S)$ the obtained N -dimensional Riemannian space. Let additionally a Finsler metric function F be introduced on this TM , yielding a Finsler space $\mathcal{F}^N = (M, F)$. We shall study the Finsler space \mathcal{F}^N can be specified according to the following definition.

INPUT DEFINITION. The Finsler space \mathcal{F}^N under consideration is the *deformed Riemannian space* \mathcal{R}^N :

$$\mathcal{F}^N = \mathbf{C} \cdot \mathcal{R}^N, \quad (\text{II.2.1})$$

specified by the condition that in each tangent space $T_x M$ the metric tensor $g_{\{x\}}(y)$ produced by the Finsler metric is the \mathbf{C} -transformation of the tensor which is *conformal* to the Euclidean metric tensor entailed by the Riemannian metric of the space \mathcal{R}^N . It is assumed that the applied \mathbf{C} -transformations (1.1) do not influence any point $x \in M$ of the base manifold M and that they are sufficiently smooth and invertible. It is also natural to require that the \mathbf{C} -transformations (1.1) send unit vectors to unit vectors:

$$\mathcal{I}F_{\{x\}} = \mathbf{C} \cdot \mathcal{S}_{\{x\}}. \quad (\text{II.2.2})$$

Additionally, we subject the \mathbf{C} -transformation to the condition of positive homogeneity with respect to tangent vectors y , denoting the degree of homogeneity by H .

If $f(x, y)$ is the involved conformal multiplier in the tangent Riemannian space $\mathcal{R}_{\{x\}}$, then the equality $g_{\{x\}}(y) = f(x, y)u_{\{x\}}(y)$ should introduce the tensor $u_{\{x\}}(y)$ which associated Riemannian curvature tensor vanishes identically. The function $f(x, y)$ is assumed naturally to be homogeneous with respect to the argument y . Denoting the homogeneity degree of $f(x, y)$ by means of $2a(x)$, we just conclude that the difference $1 - a$ is exactly the homogeneity degree of the transformation (2.1) considered, that is, $H = 1 - a$.

The following proposition is valid.

Proposition II.2.1. *A Finsler space is the \mathcal{F}^N -space if and only if the indicatrix of the Finsler space is a space of constant curvature. The dependence of the multiplier f on the variable y proves to be presented by the power of the Finsler metric function F , such that*

$$g_{\{x\}}(y) = p^2 u_{\{x\}}(y), \quad p = c_1(x) (F(x, y))^{a(x)}, \quad c_1(x) > 0. \quad (\text{II.2.3})$$

The equality $\mathcal{C}_{\text{Ind.}} = H^2$ ensues.

The proposition is of the local meaning in both the base manifold and the tangent space.

Proof. Given an arbitrary Finsler space of any dimension $N \geq 3$. The tangent Riemannian space $\mathcal{R}_{\{x\}}$ is conformally flat if and only if the indicatrix is a space of constant curvature. Indeed, in dimensions $N \geq 4$ the conformal flatness holds if and only if the respective Weyl tensor W_{ijmn} vanishes identically. By evaluating the tensor and considering the direct implications of the contraction vanishing $W_{ijmn} l^n l^j = 0$, we immediately obtain the representation $S_{nmij} = C(h_{nj}h_{mi} - h_{ni}h_{mj})$ which is characteristic of the constancy of the indicatrix curvature. In the dimension $N \geq 3$, the conformal flatness of the space $\mathcal{R}_{\{x\}}$ is tantamount to the identical vanishing of the respective Cotton-York tensor. Considering the vanishing attentively leads again to the representation $S_{nmij} = C(x)(h_{nj}h_{mi} - h_{ni}h_{mj})$. These observations prove the first part of Proposition II.2.1. All the involved computations are explicitly represented in Appendix B.

To get the required conclusions concerning the form of the respective conformal multiplier we can start with the tensor $u_{ij} = z(x, y)(c_1(x))^{-2}F^{-2a(x)}g_{ij}$, where z is a test smooth positive function homogeneous of the degree zero with respect to the argument y . We evaluate the respective curvature tensor $\tilde{R}_{\{x\}}$ and assume $\tilde{R}_{\{x\}} = 0$ to determine the tensor $S_n^m{}_{ij} = (C^h{}_{nj}C^m{}_{hi} - C^h{}_{ni}C^m{}_{hj})F^2$. After that, we consider the implications of the vanishing $S_n^m{}_{ij}l^m l^j = 0$ and arrive at the representation

$$S_{nmij} = a(2 - a)(h_{nj}h_{mi} - h_{ni}h_{mj}) + F^2 \frac{1}{2z^2} (z_h g^{hs} z_s)(h_{nj}h_{mi} - h_{ni}h_{mj})$$

$$+ \frac{a - 1}{2z} \left(z_n (l_i h_{mj} - l_j h_{mi}) - z_m (l_i h_{nj} - l_j h_{ni}) + l_n (z_i h_{mj} - z_j h_{mi}) - l_m (z_i h_{jn} - z_j h_{in}) \right) F,$$

where $z_k = \partial z / \partial y^k$. The tensor $S_n^m{}_{ij}$ must obviously possess the property $S_{nmij} l^i = 0$. Therefore, we must fulfill the equation $(a - 1)(z_n h_{mj} - z_m h_{nj}) = 0$. Because of $a \neq 1$, we can take only $z_n = 0$, which means that the function z is independent of y . Without any loss of generality we can take $z = 1$. Thus we have proved the second part in Proposition II.2.1. From the above representation of the tensor S_{nmij} we just obtain $\mathcal{C}_{\text{Ind.}} = 1 - a(2 - a) \equiv (1 - a)^2$. Since the difference $1 - a$ is equal to H , the identification $\mathcal{C}_{\text{Ind.}} = H^2$ is valid. All the computations which are required to trace the validity of the formulas exposed can be found in Appendix C. Proposition II.2.1 is valid. To have the equality $S(x, \bar{y}) = (F(x, y))^{H(x)}$, we make the choice $c_1 = 1/H$.

Let the **C**-transformation (I.2.1) proposed in Chapter I be assigned locally by means of the differentiable functions

$$\bar{y}^m = \bar{y}^m(x, y), \quad (\text{II.2.4})$$

subject to the required homogeneity

$$\bar{y}^m(x, ky) = k^H \bar{y}^m(x, y), \quad k > 0, \forall y. \quad (\text{II.2.5})$$

This entails the identity

$$\bar{y}_k^m y^k = H \bar{y}^m, \quad (\text{II.2.6})$$

where $\bar{y}_k^m = \partial \bar{y}^m / \partial y^k$. Fulfilling (2.1) means locally

$$g_{mn}(x, y) = c_{ij}(x, \bar{y}) \bar{y}_m^i \bar{y}_n^j, \quad c_{ij}(x, \bar{y}) = (p(x, y))^2 a_{ij}(x). \quad (\text{II.2.7})$$

If we contract this tensor by $y^m y^n$ and use the homogeneity identity (2.6), we obtain the equality

$$p(x, y) = \frac{1}{H(x)} \frac{F(x, y)}{S(x, \bar{y})}. \quad (\text{II.2.8})$$

On every punctured tangent space $T_x M \setminus 0$, the Finsler metric function F is assumed to be positive, and also positively homogeneous of degree 1:

$$F(x, ky) = kF(x, y), \quad k > 0, \forall y.$$

The entailed Finsler metric tensor is positively homogeneous of degree 0. Therefore, to comply the representation (2.7) with the stipulation (2.3), we must put

$$H = 1 - a. \quad (\text{II.2.9})$$

With this observation, comparing (2.3) with (2.8) yields the equality

$$c_1 S = \frac{1}{H} F^H. \quad (\text{II.2.10})$$

To comply with the indicatrix correspondence (2.2), we should put $c_1 = 1/H$, which leads to the equality $S = F^H$ indicated in (I.2.3).

Denote by

$$y^i = y^i(x, t), \quad t^n \equiv \bar{y}^n, \quad (\text{II.2.11})$$

the inverse transformation, so that

$$y^i(x, kt) = k^{1/H} y^i(x, t), \quad k > 0, \forall t,$$

and

$$y_n^i t^n = \frac{1}{H} y^i, \quad (\text{II.2.12})$$

where $y_n^i = \partial y^i / \partial t^n$. The inverse to (2.7) reads:

$$g_{kh} y_m^k y_n^h = c_{mn}. \quad (\text{II.2.13})$$

The following useful relations can readily be arrived at:

$$y_m y_n^m = \frac{F^2}{H S^2} t_n \equiv \frac{1}{H} F^{2(1-H)} t_n, \quad t_n = a_{nh} t^h, \quad (\text{II.2.14})$$

and

$$y_m y_{nl}^m t_j^l + g_{mj} y_n^m = 2 \left(\frac{1}{H} - 1 \right) F^{-2H} y_j t_n + \frac{1}{H} F^{2(1-H)} a_{nh} t_j^h,$$

where $t_j^l = \bar{y}_j^l$ and $y_{nl}^m = \partial y_n^m / \partial y^l$. Alternatively,

$$t_h t_n^h = \frac{HS^2}{F^2} y_n \equiv HF^{2(H-1)} y_n \quad (\text{II.2.15})$$

and

$$t_h t_{nu}^h y_i^u + a_{hi} t_n^h = 2(H-1)F^{-2} t_i y_n + HF^{2(H-1)} g_{nu} y_i^u, \quad (\text{II.2.16})$$

where $t_{nu}^h = \partial t_n^h / \partial y^u$. We may also write

$$t_h t_{ni}^h = H(1-H)F^{2(H-1)}(g_{ni} - 2l_n l_i). \quad (\text{II.2.17})$$

From (2.13) it follows that

$$g_{nm} y_i^m = p^2 t_n^j a_{ij}, \quad y_{hp}^k t_n^p = -y_p^k t_{nv}^p y_h^v.$$

Differentiating (2.7) with respect to y^k yields the following representation for the tensor $C_{mnk} = (1/2)\partial g_{mn}/\partial y^k$:

$$2C_{mnk} = (1-H)\frac{2}{F}l_k g_{mn} + p^2(t_{mk}^i t_n^j + t_m^i t_{nk}^j)a_{ij}. \quad (\text{II.2.18})$$

Contracting this tensor by y^n results in the equality

$$p^2 t_{mk}^i t^j a_{ij} = \left(\frac{1}{H} - 1\right)(h_{km} - l_k l_m), \quad (\text{II.2.19})$$

where the vanishing $C_{mnk} y^n = 0$ and the homogeneity identity (2.6) have been taken into account.

Symmetry of the tensor C_{mnk} demands

$$(1-H)\frac{2}{F}(l_k g_{mn} - l_m g_{kn}) + p^2(t_m^i t_{nk}^j - t_k^i t_{nm}^j)a_{ij} = 0, \quad (\text{II.2.20})$$

so that we may alternatively write

$$C_{mnk} = (1-H)\frac{1}{F}(l_k g_{mn} + l_n g_{mk} - l_m g_{nk}) + p^2 t_m^i t_{nk}^j a_{ij}. \quad (\text{II.2.21})$$

Contracting the last tensor by g^{nk} yields

$$2C_m = (1-H)\frac{2}{F}l_m + g^{nk} p^2(t_{nk}^i t_m^j + t_n^i t_{mk}^j)a_{ij} \equiv 2C_{mnk} g^{nk},$$

from which it ensues that

$$2C_m = (1-H)\frac{2}{F}l_m + 2g^{nk} p^2 t_{nk}^i t_m^j a_{ij} + g^{nk} p^2(t_n^i t_{mk}^j - t_m^i t_{nk}^j)a_{ij},$$

or

$$2C_m = (1-H)\frac{2}{F}l_m + 2g^{nk} p^2 t_{nk}^i t_m^j a_{ij} - (1-H)g^{nk} \frac{2}{F}(l_m g_{nk} - l_n g_{mk})a_{ij}.$$

It is also convenient to use the representation

$$FC_m = -(N-2)(1-H)l_m + Fg^{nk}p^2t_{nk}^it_m^ja_{ij}. \quad (\text{II.2.22})$$

Since $y_i^m = p^2t_n^ja_{ij}g^{nm}$, we can write

$$FC^m = -(N-2)(1-H)l^m + Fg^{nk}t_{nk}^iy_i^m. \quad (\text{II.2.23})$$

The space \mathcal{F}^N is obtainable from the Riemannian space \mathcal{R}^N by means of the deformation which, owing to (2.7), can be presented by the *conformal deformation tensor*

$$C_m^i := p\bar{y}_m^i, \quad (\text{II.2.24})$$

so that

$$g_{mn} = C_m^i C_n^j a_{ij}. \quad (\text{II.2.25})$$

The zero-degree homogeneity

$$C_m^i(x, ky) = C_m^i(x, y), \quad k > 0, \forall y, \quad (\text{II.2.26})$$

holds, together with

$$C_m^i(x, y)y^m = (F(x, y))^{1-H}\bar{y}^i. \quad (\text{II.2.27})$$

The indicatrix correspondence (2.2) is a direct implication of the equality $S = F^H$. We may apply the transformation (1.1) to the unit vectors:

$$l = \mathbf{C} \cdot L : \quad l^i = y^i(x, L); \quad L = \mathbf{C}^{-1} \cdot l : \quad L^i = t^i(x, l), \quad (\text{II.2.28})$$

where $l^i = y^i/F(x, y)$ and $L^i = t^i/S(x, t)$ are components of the respective Finslerian and Riemannian unit vectors, which possess the properties $F(x, l) = 1$ and $S(x, L) = 1$. We have $L^m = t^m(x, l)$. On the other hand, from (2.7) it just follows that

$$g_{mn}(x, l) = \frac{1}{H^2}a_{ij}(x)t_m^i(x, l)t_n^j(x, l), \quad (\text{II.2.29})$$

so that under the transformation (2.28) we have

$$g_{mn}(x, l)dl^m dl^n = \frac{1}{H^2}a_{ij}(x)dL^i dL^j. \quad (\text{II.2.30})$$

Note. The deformation performed by the formulas (2.24) and (2.25) is *unholonomic*, in the sense that

$$\frac{\partial C_m^i}{\partial y^n} - \frac{\partial C_n^i}{\partial y^m} \neq 0. \quad (\text{II.2.31})$$

The vanishing appears if only the factor $p = F^{1-H}/H$ is independent of the vectors y , that is, when $H = 1$ (which is the Riemannian case proper). Regarding the y -dependence, the tensor C_m^i is homogeneous of degree zero, in accordance with (2.26). If we divide the tensor by p , we obtain from (2.24) the tensor \bar{y}_m^i which is the derivative tensor, namely $\bar{y}_m^i = \partial \bar{y}^i / \partial y^m$. However, such a property cannot be addressed to the tensor C_m^i . It is the reason why we start with the stipulation that the underlined transformation (which is downloaded locally by the formulas (2.4)-(2.7)) be homogeneous of the degree H with respect to the variable y . By proceeding in this way, it proves possible to come to the

conformal representation (2.30) of $g_{mn}(x, l)dl^m dl^n$ which is of the key significance to obtain the angle and the connection coefficients.

No support vector enters the right-hand part of (2.30). Therefore, any two nonzero tangent vectors $y_1, y_2 \in T_x M$ in a fixed tangent space $T_x M$ form the \mathcal{F}^N -space angle

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)} \arccos \lambda, \quad (\text{II.2.32})$$

where the scalar

$$\lambda = \frac{a_{mn}(x)t_1^m t_2^n}{S_1 S_2}, \quad \text{with } t_1^m = t^m(x, y_1) \quad \text{and} \quad t_2^m = t^m(x, y_2), \quad (\text{II.2.33})$$

is of the entire Riemannian meaning in the space \mathcal{R}^N ; the notation $S_1 = \sqrt{a_{mn}(x)t_1^m t_1^n}$ and $S_2 = \sqrt{a_{mn}(x)t_2^m t_2^n}$ has been used.

From (2.33) it follows that

$$\begin{aligned} \frac{\partial \lambda}{\partial x^i} &= \frac{a_{mn,i} t_1^m t_2^n}{S_1 S_2} + \frac{1}{S_1 S_2} a_{mn} \left(\frac{\partial t_1^m}{\partial x^i} t_2^n + t_1^m \frac{\partial t_2^n}{\partial x^i} \right) \\ &- \frac{1}{2} \lambda \left[\frac{1}{S_1 S_1} \left(a_{mn,i} t_1^m t_1^n + 2a_{mn} \frac{\partial t_1^m}{\partial x^i} t_1^n \right) + \frac{1}{S_2 S_2} \left(a_{mn,i} t_2^m t_2^n + 2a_{mn} \frac{\partial t_2^m}{\partial x^i} t_2^n \right) \right], \end{aligned}$$

where $a_{mn,i} = \partial a_{mn} / \partial x^i$, and

$$\frac{\partial \lambda}{\partial y_1^k} = \left[\frac{a_{mn} t_2^n}{S_1 S_2} - \frac{a_{mn} t_1^n}{S_1 S_1} \lambda \right] t_{1k}^m, \quad \frac{\partial \lambda}{\partial y_2^k} = \left[\frac{a_{mn} t_1^n}{S_2 S_1} - \frac{a_{mn} t_2^n}{S_2 S_2} \lambda \right] t_{2k}^m.$$

When the recurrent preservation

$$d_i \alpha + (1/H) H_i \alpha = 0$$

proposed by (I.1.28) is applied to the angle given in (2.32), we obtain simply

$$d_i \lambda = 0, \quad (\text{II.2.34})$$

where d_i is the separable operator (I.1.11). That is, the recurrent preservation law formulated for the Finsler \mathcal{F}^N -space angle (2.32) is tantamount to the separable preservation law (2.34) for the Euclidean angle $\arccos \lambda$.

We note also that

$$A_1^k \frac{\partial \lambda}{\partial y_1^k} = F_1 g_1^{nh} t_{1nh}^i y_{1i}^k \frac{\partial \lambda}{\partial y_1^k} = F_1 g_1^{nh} t_{1nh}^i \left[\frac{t_{2i}}{S_1 S_2} - \frac{t_{1i}}{S_1 S_1} \lambda \right].$$

II.3. Derivation and properties of the coefficients N^m_n in the \mathcal{F}^N -space

Let us start from (2.11) and introduce the vector $U^i = U^i(x, y)$ according to

$$U^i \stackrel{\text{def}}{=} \frac{1}{S} t^i \equiv \frac{1}{F^H} t^i, \quad (\text{II.3.1})$$

which is obviously unit:

$$U_i U^i = 1, \quad U_i = a_{ij} U^j. \quad (\text{II.3.2})$$

The zero-degree homogeneity

$$U^i(x, ky) = U^i(x, y), \quad k > 0, \forall y, \quad (\text{II.3.3})$$

holds, entailing the identity

$$U_n^i y^n = 0, \quad (\text{II.3.4})$$

where

$$U_n^i := \frac{\partial U^i}{\partial y^n} = \frac{1}{F^H} t_n^i - \frac{1}{F} H U^i l_n. \quad (\text{II.3.5})$$

From (2.14) it follows that

$$F^H U_s^h y_h^k = h_s^k, \quad F^H U_k^i y_t^k = \delta_t^i - U^i U_t, \quad U_i U_n^i = 0. \quad (\text{II.3.6})$$

The vanishing

$$U_i \left(\frac{\partial U^i}{\partial x^n} + L_{kn}^i U^k \right) = 0 \quad (\text{II.3.7})$$

holds obviously, where $L_{kn}^i = L_{kn}^i(x)$ are the Riemannian connection coefficients appeared in (1.2).

The representation (2.33) takes on the simple form

$$\lambda = a_{mn}(x) U_1^m U_2^n, \quad (\text{II.3.8})$$

with

$$U_1^m = U^m(x, y_1), \quad U_2^m = U^m(x, y_2). \quad (\text{II.3.9})$$

The form of the right-hand part in the formula (3.8) which represents the scalar λ is such that the preservation law $d_i \lambda = 0$ written in (2.34) is obviously equivalent to the vanishing

$$\mathcal{D}_n U^i = 0 \quad (\text{II.3.10})$$

for the field $U^i = U^i(x, y)$, with the covariant derivative

$$\mathcal{D}_n U^i := d_n U^i + L_{nk}^i U^k. \quad (\text{II.3.11})$$

Since

$$d_n U^i = \frac{\partial U^i}{\partial x^n} + N_{nk}^i U^k,$$

we obtain the representation

$$N^m_n = -y_i^m F^H \left(\frac{H}{F} U^i \frac{\partial F}{\partial x^n} + \frac{\partial U^i}{\partial x^n} + L_{nk}^i U^k \right) + l^m d_n F \quad (\text{II.3.12})$$

which was indicated in (I.2.15).

We have arrived at the following proposition.

Proposition II.3.1. *Given an arbitrary smooth function $H(x)$, the angle preservation equation $d_n\alpha + (1/H)H_n\alpha = 0$ in the \mathcal{F}^N -space entails the representation (3.12) for the coefficients N^m_n .*

By differentiating (3.10) with respect to y^m we may conclude that the covariant derivative

$$\mathcal{D}_n U_m^i := d_n U_m^i - D^h_{nm} U_h^i + L^i_{nl} U_m^l, \quad D^h_{nm} = -N^h_{nm}, \quad (\text{II.3.13})$$

vanishes identically:

$$\mathcal{D}_n U_m^i = 0. \quad (\text{II.3.14})$$

Below, we shall assume that

$$d_n F = 0.$$

Using $t^i = F^H U^i$ together with

$$\mathcal{D}_n t^i := d_n t^i + L^i_{kn} t^k, \quad (\text{II.3.15})$$

from (3.10) we find

$$\mathcal{D}_n t^i = t^i H_n \ln F. \quad (\text{II.3.16})$$

Differentiating (3.16) with respect to y^m leads to the conclusion that the covariant derivative

$$\mathcal{D}_n t_m^i := d_n t_m^i - D^h_{nm} t_h^i + L^i_{nl} t_m^l \quad (\text{II.3.17})$$

possesses the property

$$\mathcal{D}_n t_m^i = \left(t_m^i \ln F + t^i l_m \frac{1}{F} \right) H_n. \quad (\text{II.3.18})$$

With $p = (1/H)F^{1-H}$ from (3.18) we get

$$\mathcal{D}_n (p t_m^i) = p t_m^i l_n \frac{1}{F} H_n - \frac{1}{H} H_n p t_m^i, \quad (\text{II.3.19})$$

so that,

$$\mathcal{D}_n (p t_m^i) = -\frac{p}{H} H_n h_m^k t_k^i. \quad (\text{II.3.20})$$

Consider (2.7):

$$g_{mn}(x, y) = p^2 t_m^i t_n^j a_{ij}(x).$$

We obtain

$$\mathcal{D}_k g_{mn} = p^2 \left(t^i l_m \frac{1}{F} - \frac{1}{H} t_m^i \right) H_k t_n^j a_{ij} + p^2 t_m^i \left(t^j l_n \frac{1}{F} - \frac{1}{H} t_n^j \right) H_k a_{ij}.$$

Using $t_h t_n^h = H F^{2(H-1)} y_n$ (see (2.34)) leads to

$$\mathcal{D}_k g_{mn} = -\frac{2}{H} H_k g_{mn} + p^2 l_m H_k H F^{2(H-1)} l_n + p^2 t_m^i t_n^j l_n \frac{1}{F} H_k a_{ij}.$$

In this way we arrive at the following result after the direct evaluations performed.

Proposition II.3.2. *Given an arbitrary smooth function $H(x)$ in the \mathcal{F}^N -space, the angle preservation $d_n\alpha + (1/H)H_n\alpha = 0$ taken in conjunction with the preservation $d_nF = 0$ of the metric function entails that the covariant derivative of the metric tensor reads*

$$\mathcal{D}_k g_{mn} = -\frac{2}{H} H_k h_{mn}. \quad (\text{II.3.21})$$

Now, we contract (3.19) by y_k^m , getting

$$y_k^m \mathcal{D}_n t_m^i = y_k^m d_n t_m^i - D^h_{nm} y_k^m t_h^i + L^i_{nk} = \left(\delta_k^i \ln F + t^i t_k \frac{1}{H F^{2H}} \right) H_n.$$

Since $y_k^n t_j^k = \delta_j^n$, the previous identity can be transformed to

$$t_m^i d_n y_k^m + D^h_{nm} y_k^m t_h^i - L^i_{nk} = - \left(\delta_k^i \ln F + t^i t_k \frac{1}{H F^{2H}} \right) H_n.$$

Contract this equality by y_i^j , obtaining the equality

$$d_n y_k^j + D^j_{nm} y_k^m - L^i_{nk} y_i^j = - \left(y_k^j \ln F + y^j t_k \frac{1}{H^2 F^{2H}} \right) H_n, \quad (\text{II.3.22})$$

which can be written simply as

$$d_n \left(\frac{1}{p} y_k^j \right) + T^j_{nm} y_k^m \frac{1}{p} - L^i_{nk} y_i^j \frac{1}{p} = 0, \quad (\text{II.3.23})$$

where T^j_{nm} are the coefficients introduced in (I.1.33). Taking into account the representations (3.15)-(3.16) together with the identity

$$t^m \frac{\partial}{\partial t^m} \left(\frac{1}{p} y_k^j \right) = 0$$

ensuing from the homogeneity, the equality (3.23) becomes

$$d_n^{\text{Riem}} \left(\frac{1}{p} y_k^j \right) + T^j_{nm} y_k^m \frac{1}{p} - L^i_{nk} y_i^j \frac{1}{p} = 0. \quad (\text{II.3.24})$$

We have used the Riemannian operator d_n^{Riem} introduced in (II.1.4).

We know that

$$d_n^{\text{Riem}} t^k + L^k_{nm} t^m = 0. \quad (\text{II.3.25})$$

Therefore, contracting (3.24) by t^k yields

$$d_n^{\text{Riem}} \left(\frac{1}{H p} y^j \right) - N^j_n \frac{1}{H p} = 0. \quad (\text{II.3.26})$$

Here we have

$$Hp = F^{1-H} = S^{1/H} \frac{1}{S}, \quad (\text{II.3.27})$$

so that

$$d_n^{\text{Riem}} \frac{1}{Hp} = \frac{1}{Hp} \frac{1}{H^2} H_n \ln S \equiv \frac{1}{Hp} \frac{1}{H} H_n \ln F. \quad (\text{II.3.28})$$

We arrive at the following proposition.

Proposition II.3.3. *With an arbitrary smooth function $H(x)$, in the \mathcal{F}^N -space with $d_n F = 0$ the representation*

$$N^m_n = d_n^{\text{Riem}} y^m(x, t) + \frac{1}{H} H_n y^m \ln F \quad (\text{II.3.29})$$

written by the help of the Riemannian operator d_n^{Riem} is valid.

The derivative coefficients $N^k_{mn} = \partial N^k_m / \partial y^n$ can straightforwardly be evaluated from the coefficients N^k_m written in (I.2.16). We obtain

$$\begin{aligned} N^k_{mn} = & -\frac{1}{F} h_n^k \frac{\partial F}{\partial x^m} - l^k \frac{\partial l_n}{\partial x^m} - \left(y_{hp}^k t_n^p + \frac{H}{F} y_h^k l_n \right) F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U^s \right) \\ & - y_h^k F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U_n^s \right). \end{aligned}$$

Owing to (3.10)), we have

$$\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U^s = -N^s_m U_s^h,$$

so that using (3.6) we observe that the coefficients N^k_{mn} are equal to

$$-\frac{1}{F} h_n^k \frac{\partial F}{\partial x^m} - l^k \frac{\partial l_n}{\partial x^m} + U_s^h y_{hp}^k t_n^p F^H N^s_m + \frac{H}{F} y_h^k l_n F^H N^s_m U_s^h - y_h^k F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U_n^s \right),$$

or

$$N^k_{mn} = -\frac{1}{F} h_n^k \frac{\partial F}{\partial x^m} - l^k \frac{\partial l_n}{\partial x^m} - h_s^v y_j^k t_{nv}^j N^s_m + \frac{H}{F} h_s^k l_n N^s_m - y_h^k F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U_n^s \right),$$

where the relation

$$F^H U_s^h y_{hp}^k t_n^p = -F^H U_s^h y_p^k t_{nv}^p y_h^v = -h_s^v y_p^k t_{nv}^p$$

has been used.

From (2.21) we have

$$p^2 t_m^i t_{nk}^j a_{ij} = C_{mnk} - (1 - H) \frac{1}{F} (l_k g_{mn} + l_n g_{mk} - l_m g_{nk}),$$

which is

$$g_{mv}y_j^v t_{nk}^j = C_{mnk} - (1-H)\frac{1}{F}(l_k g_{mn} + l_n g_{mk} - l_m g_{nk}).$$

We obtain

$$y_j^k t_{nv}^j = C_{nv}^k - (1-H)\frac{1}{F}(l_v \delta_n^k + l_n \delta_v^k - l^k g_{nv}) \quad (\text{II.3.30})$$

and

$$h_s^v y_j^k t_{nv}^j = C_{ns}^k - (1-H)\frac{1}{F}(l_n h_s^k - l^k h_{ns}). \quad (\text{II.3.31})$$

In this way we come to the representation

$$\begin{aligned} N^k_{mn} = & -\frac{1}{F}h_n^k \frac{\partial F}{\partial x^m} - l^k \frac{\partial l_n}{\partial x^m} - \left(C_{ns}^k - (1-H)\frac{1}{F}(l_n h_s^k - l^k h_{ns}) \right) N^s_m + \frac{H}{F}h_s^k l_n N^s_m \\ & - y_h^k F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U_n^s \right). \end{aligned}$$

The eventual result reads

$$\begin{aligned} N^k_{mn} = & -\frac{1}{F}h_n^k \frac{\partial F}{\partial x^m} - l^k \frac{\partial l_n}{\partial x^m} - C_{ns}^k N^s_m + \frac{1}{F}(l_n h_s^k - (1-H)l^k h_{ns}) N^s_m \\ & - y_h^k F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U_n^s \right). \end{aligned} \quad (\text{II.3.32})$$

Thus we can formulate the following assertion.

Proposition II.3.4. *With an arbitrary smooth function $H(x)$, in the \mathcal{F}^N -space with $d_n F = 0$ the coefficients N^k_{mn} can be given by means of the explicit representation written in (3.32).*

We are also able to evaluate the entailed coefficients $N^k_{mni} = \partial N^k_{mn} / \partial y^i$. The required evaluations which have been presented in detail in Appendix D lead straightforwardly to the representation

$$N^k_{mni} = \frac{2}{H}H_m \frac{1}{F}l^k h_{ni} - \mathcal{D}_m C^k_{ni}. \quad (\text{II.3.33})$$

Thus we can formulate the following assertion.

Proposition II.3.5. *Given an arbitrary smooth function $H(x)$, in the \mathcal{F}^N -space with $d_n F = 0$ the coefficients N^k_{mni} admit the simple representation (3.33) in terms of the covariant derivative of the tensor C^k_{ni} .*

II.4. Properties of covariant derivative

The equality (3.20) can be written in the form

$$\mathcal{T}_i(pt_n^m) = 0 \quad (\text{II.4.1})$$

with

$$\mathcal{T}_i(pt_n^m) = d_i(pt_n^m) - T^h_{in}pt_h^m + L^m_{il}pt_n^l, \quad (\text{II.4.2})$$

where T^h_{in} are the coefficients (I.1.24). If we contract the last vanishing by y^n and note that $\mathcal{T}_iy^n = 0$ (see (I.1.37)), we get

$$\mathcal{T}_i(Hpt^m) = 0, \quad (\text{II.4.3})$$

where

$$\mathcal{T}_i(Hpt^m) = d_i(Hpt^m) + L^m_{ik}Hpt^k. \quad (\text{II.4.4})$$

We may write

$$\mathcal{T}_i(Hpt^m) = t^m d_i(Hp) + Hp\mathcal{D}_it^m, \quad (\text{II.4.5})$$

where $\mathcal{D}_it^m = d_it^m + L^m_{ik}t^k$.

Owing to the equality $C_n^m = pt_n^m$ (see (2.24)), from (4.1) we are entitled to formulate the following proposition.

Proposition II.4.1. *The \mathbf{C} -deformation is \mathcal{T} -covariant constant:*

$$\mathcal{T} \cdot \mathbf{C} = 0. \quad (\text{II.4.6})$$

In terms of local coordinates the previous vanishing reads

$$\mathcal{T}_nC_k^m = 0, \quad (\text{II.4.7})$$

where

$$\mathcal{T}_nC_k^m = d_nC_k^m - T^h_{nk}C_h^m + L^m_{nl}C_k^l. \quad (\text{II.4.8})$$

The reciprocal coefficients

$$\tilde{C}_m^n = \frac{1}{p}y_m^n \quad (\text{II.4.9})$$

fulfills the similar vanishing

$$\mathcal{T}_n\tilde{C}_k^m = 0, \quad (\text{II.4.10})$$

where

$$\mathcal{T}_n\tilde{C}_k^m = d_n^{\text{Riem}}\tilde{C}_k^m + T^m_{nh}\tilde{C}_k^h - L^i_{nk}\tilde{C}_i^m \quad (\text{II.4.11})$$

(see (3.23)).

Let us realize the action of the \mathcal{C} -transformation (2.1)-(2.2) on tensors by the help of the deformation

$$\{w(x, y)\} = \mathcal{C} \cdot \{W(x, t)\}, \quad (\text{II.4.12})$$

assuming that the tensors $\{w(x, y)\}$ are positively homogeneous of degree 0 with respect

to the variable y , and that the tensors $\{W(x, t)\}$ are positively homogeneous of degree 0 with respect to the variable t . Namely, in the scalar case we use the identification

$$w(x, y) = W(x, t), \quad (\text{II.4.13})$$

obtaining merely

$$d_i w = d_i^{\text{Riem}} W \quad (\text{II.4.14})$$

(because of the vanishing $\mathcal{D}_i U^j = 0$ indicated in (3.10)-(3.11)), where d_i^{Riem} is the operator defined by (II.1.4). Given a tensor $w_n(x, y)$ of the type (0,1) we use the transformation

$$w_n = C_n^m W_m. \quad (\text{II.4.15})$$

The metrical linear connection $\mathcal{R}L$ introduced by (1.2) may be used to define the covariant derivative ∇ in \mathcal{R}^N according to the conventional rule:

$$\nabla_i W_m = \frac{\partial W_m}{\partial x^i} + L^k_i \frac{\partial W_m}{\partial t^k} - L^h_{im} W_h, \quad L^k_j = -L^k_{ij} t^i, \quad (\text{II.4.16})$$

which can be written shortly

$$\nabla_i W_m = d_i^{\text{Riem}} W_m - L^h_{im} W_h. \quad (\text{II.4.17})$$

We have

$$\nabla_i S = 0, \quad \nabla_i t^j = 0, \quad \nabla_i a_{mn} = 0. \quad (\text{II.4.18})$$

By virtue of the nullification $\mathcal{T}_i C_n^m = 0$ shown in (4.7), we obtain the *transitivity property*

$$\mathcal{T}_i w_n = C_n^m \nabla_i W_m \quad (\text{II.4.19})$$

for the covariant derivatives.

The method can be repeated in case of the covariant vectors $w^n(x, y)$ and $W^n(x, t)$, namely we write

$$w^n = \tilde{C}_m^n W^m, \quad (\text{II.4.20})$$

obtaining

$$\mathcal{T}_i w^n = \tilde{C}_m^n \nabla_i W^m, \quad (\text{II.4.21})$$

where the reciprocal coefficients $\tilde{C}_m^n = (1/p)y_m^n$ defined by (4.9) have been arisen.

The method can also be extended to more general tensors in a direct manner. For example, considering the (1,1)-type tensors $\{w_m^n(x, y), W_m^n(x, t)\}$ of the zero-degree positive homogeneity with respect to the variables y and t , we can use the covariant derivative

$$\nabla_i W_m^n = \frac{\partial W_m^n}{\partial x^i} + L^k_i \frac{\partial W_m^n}{\partial t^k} + L^n_{hi} W_m^h - L^h_{mi} W_h^n \equiv d_i^{\text{Riem}} W_m^n + L^n_{hi} W_m^h - L^h_{mi} W_h^n \quad (\text{II.4.22})$$

and the deformation

$$w_m^n = \tilde{C}_h^n C_m^k W_k^h \quad (\text{II.4.23})$$

to obtain the transitivity property

$$\mathcal{T}_i w_m^n = \tilde{C}_h^n C_m^k \nabla_i W_k^h \quad (\text{II.4.24})$$

for the covariant derivatives \mathcal{T}_i and ∇_i .

Now we may formulate the following proposition.

Proposition II.4.2. *The covariant derivative \mathcal{T} is the manifestation of the transitivity of the connection under the \mathcal{C} -transformation.*

In short,

$$\mathcal{T} = \mathcal{C} \cdot \nabla. \quad (\text{II.4.25})$$

II.5. Entailed curvature tensor

Henceforth, the torsion tensor S^m_{ij} (entered the initial connection (1.2)) is not accounted for.

Given a tensor $w^n_k = w^n_k(x, y)$ of the tensorial type (1,1), commuting the covariant derivative

$$\mathcal{T}_i w^n_k := d_i w^n_k + T^n_{ih} w^h_k - T^h_{ik} w^n_h \quad (\text{II.5.1})$$

yields the equality

$$[\mathcal{T}_i \mathcal{T}_j - \mathcal{T}_j \mathcal{T}_i] w^n_k = M^h_{ij} \frac{\partial w^n_k}{\partial y^h} - E_k^h{}_{ij} w^n_h + E_h^n{}_{ij} w^h_k \quad (\text{II.5.2})$$

with the tensors

$$M^n_{ij} := d_i N^n_j - d_j N^n_i \quad (\text{II.5.3})$$

and

$$E_k^n{}_{ij} := d_i T^n_{jk} - d_j T^n_{ik} + T^m_{jk} T^n_{im} - T^m_{ik} T^n_{jm}. \quad (\text{II.5.4})$$

By applying the commutation rule (5.2) to the particular choices $\{F, y^n, y_k, g_{nk}\}$ and noting the vanishing $\{\mathcal{T}_i F = \mathcal{T}_i y^n = \mathcal{T}_i y_k = \mathcal{T}_i g_{nk} = 0\}$, we obtain the identities

$$y_n M^n_{ij} = 0, \quad y^k E_k^n{}_{ij} = -M^n_{ij}, \quad y_n E_k^n{}_{ij} = M_{kij}, \quad (\text{II.5.5})$$

and

$$E_{mni} + E_{nmi} = 2C_{mnh} M^h_{ij} \quad \text{with} \quad C_{mnh} = \frac{1}{2} \frac{\partial g_{mn}}{\partial y^h}. \quad (\text{II.5.6})$$

It proves pertinent to replace in the commutator (5.2) the partial derivative $\partial w^n_k / \partial y^h$ by the definition

$$\mathcal{S}_h w^n_k := \frac{\partial w^n_k}{\partial y^h} + C^m_{hs} w^s_k - C^m_{hk} w^n_m \quad (\text{II.5.7})$$

which has the meaning of the covariant derivative in the tangent space supported by the point $x \in M$. In particular,

$$\mathcal{S}_h g_{nk} := \frac{\partial g_{nk}}{\partial y^h} - C^m_{hn} g_{mk} - C^m_{hk} g_{nm} = 0.$$

With the *curvature tensor*

$$\rho_k^n{}_{ij} := E_k^n{}_{ij} - M^h_{ij} C^m_{hk}, \quad (\text{II.5.8})$$

the commutator (5.2) takes on the form

$$(\mathcal{T}_i \mathcal{T}_j - \mathcal{T}_j \mathcal{T}_i) w^n_k = M^h_{ij} \mathcal{S}_h w^n_k - \rho_k^h{}_{ij} w^n_h + \rho_h^n{}_{ij} w^h_k. \quad (\text{II.5.9})$$

We denote $\rho_{knij} = g_{mn}\rho_k^m{}_{ij}$. The skew-symmetry

$$\rho_{mnij} = -\rho_{nmij} \quad (\text{II.5.10})$$

holds (cf. (5.6)).

The equalities

$$y^k \rho_k^{n}{}_{ij} = -M^n{}_{ij}, \quad y_n \rho_k^{n}{}_{ij} = M_{kij}, \quad (\text{II.5.11})$$

obviously hold.

Let us evaluate the tensor $M^n{}_{ij}$, using the coefficients

$$N^n{}_j = -l^n \frac{\partial F}{\partial x^j} - y_h^n F^H \left(\frac{\partial U^h}{\partial x^j} + L^h{}_{kj} U^k \right)$$

indicated in (I.2.16).

We directly obtain

$$\begin{aligned} M^n{}_{ij} = & -\frac{1}{F} N^n{}_i \frac{\partial F}{\partial x^j} + \frac{1}{F} N^n{}_j \frac{\partial F}{\partial x^i} - l^n N^s{}_i \frac{\partial l_s}{\partial x^j} + l^n N^s{}_j \frac{\partial l_s}{\partial x^i} \\ & - d_i (y_h^n F^H) \left(\frac{\partial U^h}{\partial x^j} + L^h{}_{kj} U^k \right) + d_j (y_h^n F^H) \left(\frac{\partial U^h}{\partial x^i} + L^h{}_{ki} U^k \right) \\ & - y_h^n F^H \left[\left(\frac{\partial L^h{}_{kj}}{\partial x^i} - \frac{\partial L^h{}_{ki}}{\partial x^j} \right) U^k + L^h{}_{kj} \frac{\partial U^k}{\partial x^i} - L^h{}_{ki} \frac{\partial U^k}{\partial x^j} \right] \\ & - y_h^n F^H \left[N^s{}_i \left(\frac{\partial U_s^h}{\partial x^j} + L^h{}_{kj} U_s^k \right) - N^s{}_j \left(\frac{\partial U_s^h}{\partial x^i} + L^h{}_{ki} U_s^k \right) \right]. \end{aligned}$$

Using here the Riemannian curvature tensor

$$a_k{}^h{}_{ij} = \frac{\partial L^h{}_{kj}}{\partial x^i} - \frac{\partial L^h{}_{ki}}{\partial x^j} + L^u{}_{kj} L^h{}_{ui} - L^u{}_{ki} L^h{}_{uj} \quad (\text{II.5.12})$$

leads to

$$\begin{aligned} M^n{}_{ij} = & -\frac{1}{F} N^n{}_i \frac{\partial F}{\partial x^j} + \frac{1}{F} N^n{}_j \frac{\partial F}{\partial x^i} - l^n N^s{}_i \frac{\partial l_s}{\partial x^j} + l^n N^s{}_j \frac{\partial l_s}{\partial x^i} \\ & - d_i (y_h^n F^H) \left(\frac{\partial U^h}{\partial x^j} + L^h{}_{kj} U^k \right) + d_j (y_h^n F^H) \left(\frac{\partial U^h}{\partial x^i} + L^h{}_{ki} U^k \right) \\ & - y_h^n F^H \left[(a_k{}^h{}_{ij} - L^u{}_{kj} L^h{}_{ui} + L^u{}_{ki} L^h{}_{uj}) U^k + L^h{}_{kj} \frac{\partial U^k}{\partial x^i} - L^h{}_{ki} \frac{\partial U^k}{\partial x^j} \right] \\ & - y_h^n F^H \left[N^s{}_i (d_j U_s^h + L^h{}_{kj} U_s^k) - N^s{}_j (d_i U_s^h + L^h{}_{ki} U_s^k) \right]. \end{aligned}$$

Now, we apply the vanishing $d_i U^h + L^h_{is} U^s = 0$ (see (3.10)-(3.11)), getting

$$\begin{aligned}
M^n_{ij} = & -\frac{1}{F} N^n_i \frac{\partial F}{\partial x^j} + \frac{1}{F} N^n_j \frac{\partial F}{\partial x^i} - l^n N^s_i \frac{\partial l_s}{\partial x^j} + l^n N^s_j \frac{\partial l_s}{\partial x^i} \\
& + d_i (y_h^n F^H) N^t_j U_t^h - d_j (y_h^n F^H) N^t_i U_t^h \\
& - y_h^n F^H (a_k^h{}_{ij} U^k - L^h{}_{kj} N^t_i U_t^k + L^h{}_{ki} N^t_j U_t^k) \\
& - y_h^n F^H [N^t_i (d_j U_t^h + L^h{}_{kj} U_t^k) - N^t_j (d_i U_t^h + L^h{}_{ki} U_t^k)].
\end{aligned}$$

Taking into account the equality

$$F^H U_t^h y_h^k = h_t^k$$

(see (3.6)), we arrive at the representation

$$\begin{aligned}
M^n_{ij} = & -\frac{1}{F} N^n_i \frac{\partial F}{\partial x^j} + \frac{1}{F} N^n_j \frac{\partial F}{\partial x^i} - l^n N^s_i \frac{\partial l_s}{\partial x^j} + l^n N^s_j \frac{\partial l_s}{\partial x^i} \\
& - y_h^n F^H a_k^h{}_{ij} U^k - N^t_i d_j h_t^n + N^t_j d_i h_t^n,
\end{aligned}$$

which can readily be simplified to read

$$\begin{aligned}
M^n_{ij} = & \frac{1}{F} N^n_i N^s_j l_s - \frac{1}{F} N^n_j N^s_i l_s - l^n N^s_i \frac{\partial l_s}{\partial x^j} + l^n N^s_j \frac{\partial l_s}{\partial x^i} \\
& - y_h^n F^H a_k^h{}_{ij} U^k + N^s_i d_j (l^n l_s) - N^s_j d_i (l^n l_s) \\
= & -l^n N^s_i \frac{\partial l_s}{\partial x^j} + l^n N^s_j \frac{\partial l_s}{\partial x^i} - y_h^n F^H a_k^h{}_{ij} U^k + l^n N^s_i d_j l_s - l^n N^s_j d_i l_s.
\end{aligned}$$

The eventual result is

$$M^n_{ij} = -y_t^n t^h a_h{}^t{}_{ij}. \quad (\text{II.5.13})$$

Next, we use the equaity

$$\mathcal{T}_i \mathcal{T}_j w^n{}_m = \tilde{C}_h^n C_m^k \nabla_i \nabla_j W^h{}_k$$

(see (4.24)) to consider the relation

$$[\mathcal{T}_i \mathcal{T}_j - \mathcal{T}_j \mathcal{T}_i] w^n{}_m = \tilde{C}_h^n C_m^k [\nabla_i \nabla_j - \nabla_j \nabla_i] W^h{}_k.$$

In the commutator

$$[\nabla_i \nabla_j - \nabla_j \nabla_i] W^n_k = -t^s a_s^h{}_{ij} \frac{\partial W^n_k}{\partial t^h} - a_k^h{}_{ij} W^n_h + a_h^n{}_{ij} W^h_k \quad (\text{II.5.14})$$

the Riemannian curvature tensor $a_s^h{}_{ij}$ is constructed in accordance with the ordinary rule (5.12). Whence,

$$M^h{}_{ij} \frac{\partial w^n_m}{\partial y^h} - E_m^h{}_{ij} w^n_h + E_h^n{}_{ij} w^h_m = \tilde{C}_h^m C_m^k \left[-t^s a_s^r{}_{ij} \frac{\partial W^h_k}{\partial t^r} - a_k^r{}_{ij} W^h_r + a_r^h{}_{ij} W^r_k \right].$$

Using here the equality

$$W^h_k = C_n^h \tilde{C}_k^m w^n_m$$

(taken from (4.23)) leads to

$$M^h{}_{ij} \frac{\partial w^n_m}{\partial y^h} - E_m^h{}_{ij} w^n_h + E_h^n{}_{ij} w^h_m = -t^s a_s^h{}_{ij} \frac{\partial w^n_m}{\partial t^h} + W^h_k t^s a_s^r{}_{ij} \frac{\partial y_h^n t_m^k}{\partial t^r}$$

$$-t_m^k a_k^r{}_{ij} y_r^h w^n_h + y_s^n a_r^s{}_{ij} t_h^r w^h_m$$

Now we use here the representation (5.13) obtained for the tensor $M^h{}_{ij}$. We are left with

$$-E_m^h{}_{ij} w^n_h + E_h^n{}_{ij} w^h_m = t_u^h y_k^v w^u{}_v t^s a_s^r{}_{ij} \frac{\partial y_h^n t_m^k}{\partial t^r} - t_m^k a_k^r{}_{ij} y_r^h w^n_h + y_s^n a_r^s{}_{ij} t_h^r w^h_m.$$

In this way we obtain the explicit representation

$$E_k^n{}_{ij} = y_h^n t_{km}^h M^m{}_{ij} + y_m^n a_h^m{}_{ij} t_k^h. \quad (\text{II.5.15})$$

From (5.8) and (5.15) it follows that

$$\rho_k^n{}_{ij} = (y_h^n t_{km}^h - C_{mk}^n) M^m{}_{ij} + y_m^n a_h^m{}_{ij} t_k^h.$$

Inserting here the tensor C_{mk}^n taken from (2.21) and noting the vanishing $l_m M^m{}_{ij} = 0$ (see (5.5)), we get

$$\rho_k^n{}_{ij} = \left(y_h^n t_{km}^h - (1-H) \frac{1}{F} (l_k \delta_m^n + l^n g_{mk}) - p^2 t_m^l t_{rk}^h a_{lh} g^{nr} \right) M^m{}_{ij} + y_m^n a_h^m{}_{ij} t_k^h.$$

Let us lower here the index n and use the equality $g_{nm} y_i^m = p^2 t_n^j a_{ij}$ (see the formulas below (2.17)). This yields

$$\rho_{kni} = \left(p^2 t_n^l t_{km}^h a_{lh} - (1-H) \frac{1}{F} (l_k g_{mn} + l_n g_{mk}) - p^2 t_m^l t_{nk}^h a_{lh} \right) M^m{}_{ij} + p^2 a_{ml} t_n^l a_h^m{}_{ij} t_k^h.$$

Next, we use here the skew-symmetry relation (2.20), obtaining

$$\rho_{kni} = \left((1-H) \frac{2}{F} (l_n g_{mk} - l_m g_{kn}) - (1-H) \frac{1}{F} (l_k g_{mn} + l_n g_{mk}) \right) M^m{}_{ij} + p^2 a_{ml} t_n^l a_h^m{}_{ij} t_k^h,$$

or

$$\rho_{knij} = -(1-H)\frac{1}{F}(l_k M_{nij} - l_n M_{kij}) + p^2 a_{hlij} t_k^h t_n^l, \quad (\text{II.5.16})$$

where $a_{hlij} = a_{lr} a_h^r{}_{ij}$. Finally, we return the index n to the upper position, arriving at

$$\rho_k{}^n{}_{ij} = -(1-H)\frac{1}{F}(l_k \delta_m^n - l^n g_{mk}) M^m{}_{ij} + y_m^n a_h^m{}_{ij} t_k^h. \quad (\text{II.5.17})$$

The totally contravariant components

$$\rho^{knij} := g^{pk} a^{mi} a^{nj} \rho_p{}^n{}_{mn}$$

read

$$\rho^{knij} = -(1-H)\frac{1}{F}(l^k M^{nij} - l^n M^{kij}) + \frac{1}{p^2} y_h^k y_r^n a^{hrij}, \quad (\text{II.5.18})$$

where $a^{hrij} = a^{hl} a^{mi} a^{nj} a_l^r{}_{mn}$ and $M^{mij} := a^{hi} a^{nj} M^m{}_{hn}$.

Similarly, we can conclude from (5.13) that the tensor $M_{nij} := g_{nm} M^m{}_{ij}$ can be given by means of the representation

$$M_{nij} = -p^2 t^h t_n^m a_{hmij}. \quad (\text{II.5.19})$$

Squaring yields

$$M^{nij} M_{nij} = p^2 t^l a_l^{nij} t^h a_{hnij}. \quad (\text{II.5.20})$$

Now we square the ρ -tensor:

$$\rho^{knij} \rho_{knij} = (1-H)^2 \frac{2}{F^2} M^{nij} M_{nij} - 2(1-H) \frac{1}{F} (l^k M^{nij} - l^n M^{kij}) p^2 a_{hlij} t_k^h t_n^l + a^{knij} a_{knij}$$

$$= (1-H)^2 \frac{2}{F^2} M^{nij} M_{nij} - 2(1-H) H \frac{1}{F^2} p^2 (a_{hlij} t^h t_n^l M^{nij} - a_{hlij} t_k^h t^l M^{kij}) + a^{knij} a_{knij},$$

or

$$\rho^{knij} \rho_{knij} = (1-H)^2 \frac{2p^2}{F^2} t^l a_l^{nij} t^h a_{hnij} + 2(1-H) \frac{H p^2}{F^2} (a_{hlij} t^h t^r a_r^{lij} - a_{hlij} t^l t^r a_r^{hij}) + a^{knij} a_{knij},$$

which is

$$\rho^{knij} \rho_{knij} = a^{knij} a_{knij} + \frac{2}{S^2} \left(\frac{1}{H^2} - 1 \right) t^l a_l^{nij} t^h a_{hnij}. \quad (\text{II.5.21})$$

Because of the nullifications

$$\mathcal{T}_i \left(\frac{1}{p} y_m^n \right) = 0, \quad \mathcal{T}_i (H p t^m) = 0$$

(see (4.3) and (4.10)), from (5.13) it follows that

$$\mathcal{T}_l M^n{}_{ij} = -y_t^n t^h \left(\nabla_l - \frac{1}{H} H_l \right) a_h{}^t{}_{ij}. \quad (\text{II.5.22})$$

From (5.17) we can conclude that

$$\begin{aligned} \mathcal{T}_l \rho_k^n{}_{ij} &= (1 - H) \frac{1}{F} (l_k \delta_m^n - l^n g_{mk}) y_t^{m t h} \left(\nabla_l - \frac{1}{H} H_l \right) a_h^t{}_{ij} + y_m^n t_k^h \nabla_l a_h^m{}_{ij} \\ &\quad + H_l \frac{1}{F} (l_k \delta_m^n - l^n g_{mk}) M^m{}_{ij}. \end{aligned} \quad (\text{II.5.23})$$

The covariant derivatives

$$\mathcal{T}_k M^n{}_{ij} = d_k M^n{}_{ij} + T^n{}_{kt} M^t{}_{ij} - a^s{}_{ki} M^n{}_{sj} - a^s{}_{kj} M^n{}_{is} \quad (\text{II.5.24})$$

and

$$\mathcal{T}_l \rho_k^n{}_{ij} = d_l \rho_k^n{}_{ij} + T^n{}_{lt} \rho_k^t{}_{ij} - T^t{}_{lk} \rho_t^n{}_{ij} - a^s{}_{li} \rho_k^n{}_{sj} - a^s{}_{lj} \rho_k^n{}_{is} \quad (\text{II.5.25})$$

have been used.

Appendix A: Evaluations for Finsleroid connection coefficients with $g = g(x)$

Below we present various important evaluations which underlined the consideration performed in Section I.3 of Chapter I.

We shall use the relations

$$\frac{\partial \frac{b}{q}}{\partial y^n} = \frac{2B}{NK g q^2} A_n \quad (\text{A.1})$$

and

$$\frac{\partial \frac{q^2}{B}}{\partial y^n} = -\frac{q^2}{B} \frac{q^2}{B} \left[\frac{2B}{NK q^2} A_n + 2 \frac{2B}{NK g q^2} A_n \frac{b}{q} \right] = -\frac{q^2}{B} \frac{2}{NK} \left(1 + \frac{2b}{gq} \right) A_n,$$

so that

$$\frac{\partial \frac{q^2}{B}}{\partial y^n} = -\frac{q}{B} \frac{2}{gNK} (2b + gq) A_n. \quad (\text{A.2})$$

Moreover,

$$\frac{\partial \frac{bq}{B}}{\partial y^n} = -\frac{q}{B} \frac{2}{gNK} (2b + gq) A_n \frac{b}{q} + \frac{q^2}{B} \frac{2B}{NK g q^2} A_n$$

and

$$\frac{\partial \frac{bq}{B}}{\partial y^n} = -\frac{b}{B} (2b + gq) \frac{2}{gNK} A_n + \frac{2}{gNK} A_n. \quad (\text{A.3})$$

We shall also meet the convenience to apply the identity

$$(2b + gq) \left(q + \frac{1}{2}gb \right) = 2h^2bq + gB. \quad (\text{A.4})$$

The equality

$$\frac{\partial \bar{M}}{\partial y^n} = 2 \frac{q^2}{B} \frac{2}{gNK} A_n \quad (\text{A.5})$$

can be obtained from the relation

$$\frac{\partial y_n}{\partial g} = \bar{M} y_n + \frac{1}{2} K^2 \frac{\partial \bar{M}}{\partial y^n}. \quad (\text{A.6})$$

It follows that

$$\begin{aligned} \frac{\partial g_{mn}}{\partial g} &= \bar{M} g_{mn} + 2 \frac{q^2}{B} \frac{2}{gNK} A_m y_n + y_m \frac{q^2}{B} \frac{2}{gNK} A_n \\ &\quad - \frac{q}{B} \frac{2}{gN} (2b + gq) A_m \frac{2}{gN} A_n + \frac{q^2}{B} \frac{2}{gN} \left[-A_m l_n + \frac{2}{N} A_m A_n - \frac{gN}{2} \frac{b}{q} \mathcal{H}_{mn} \right], \end{aligned}$$

which is

$$\frac{\partial g_{mn}}{\partial g} = \bar{M} g_{mn} + \frac{q^2}{B} \frac{2}{gN} (A_m l_n + A_n l_m) - \frac{bq}{B} \frac{2}{gN} \frac{2}{gN} A_m A_n - \frac{bq}{B} h_{mn}, \quad (\text{A.7})$$

entailing

$$\frac{\partial h_{mn}}{\partial g} = -\frac{bq}{B} \frac{2}{gN} \frac{2}{gN} A_m A_n - \frac{bq}{B} h_{mn}. \quad (\text{A.8})$$

We can also obtain

$$\begin{aligned} \frac{\partial A_{mnj}}{\partial g} &= \frac{3}{2} \bar{M} A_{mnj} + \left(\frac{1}{g} - \frac{bq}{B} \right) A_{mnj} - \frac{gbq}{B} \frac{1}{gN} (A_m h_{nj} + A_n h_{mj} + A_j h_{mn}) \\ &\quad - \frac{gbq}{B} \frac{1}{gN} \frac{2}{gN} \frac{2}{gN} A_j A_m A_n, \end{aligned}$$

or

$$\frac{\partial A_{mnj}}{\partial g} = \frac{3}{2} \bar{M} A_{mnj} + \left(\frac{1}{g} - 2 \frac{bq}{B} \right) A_{mnj} - 2 \frac{gbq}{B} \frac{1}{gN} \frac{2}{gN} \frac{2}{gN} A_j A_m A_n. \quad (\text{A.9})$$

Evaluations frequently involve the vector $m_i = (2/Ng)A_i$ which possesses the properties

$$g^{ij} m_i m_j = 1, \quad y^i m_i = 0.$$

From (A.24) of [7] it follows that

$$m_i = K \frac{1}{q} (b_i - \frac{b}{K^2} y_i). \quad (\text{A.10})$$

The equality

$$K \frac{\partial m_i}{\partial y^n} = -m_n l_i + g m_n m_i - \frac{b}{q} \mathcal{H}_{in} \quad (\text{A.11})$$

holds, where $\mathcal{H}_{in} = h_{in} - m_i m_n$.

The contravariant components m^i can be taken from (A.27) of [7]:

$$m^i = \frac{1}{qK} \left[q^2 b^i - (b + gq) v^i \right], \quad (\text{A.12})$$

entailing

$$K \frac{\partial m^i}{\partial y^n} = -m_n l^i - g m^i m_n - \frac{1}{q} (b + gq) \mathcal{H}_n^i. \quad (\text{A.13})$$

With the representation

$$A_{ijk} = \frac{1}{N} \left[A_i h_{jk} + A_j h_{ik} + A_k h_{ij} - \frac{4}{N^2 g^2} A_i A_j A_k \right] \quad (\text{A.14})$$

(see (A.8) in [7]), we find that

$$\frac{\partial A_m A^k}{\partial y^n} = \frac{1}{K} A^k \left[-A_n l_m - \frac{Ng}{2} \frac{b}{q} \mathcal{H}_{mn} \right] + \frac{1}{K} A_m \left[-A_n l^k - \frac{Ng}{2} \frac{1}{q} (b + gq) \mathcal{H}_n^k \right]. \quad (\text{A.15})$$

Recollecting the scalar $h(x) = \sqrt{1 - (g^2(x)/4)}$ and introducing the scalar $G = g/h$, we get

$$\frac{\partial h}{\partial g} = -\frac{1}{4} G, \quad \frac{\partial G}{\partial g} = \frac{1}{h^3}, \quad (\text{A.16})$$

so that

$$\frac{\partial K^2}{\partial g} = \bar{M} K^2, \quad (\text{A.17})$$

where

$$\bar{M} = \frac{bq}{B} - \frac{1}{h^3} f + \frac{1}{2} \frac{G}{hB} (q^2 + \frac{1}{2} g b q),$$

or

$$\bar{M} = -\frac{1}{h^3} f + \frac{1}{2} \frac{G}{hB} q^2 + \frac{1}{h^2 B} b q. \quad (\text{A.18})$$

The function $K(x, y)$ is given by the formulas

$$K(x, y) = \sqrt{B(x, y)} J(x, y), \quad J(x, y) = e^{-\frac{1}{2} G(x) f(x, y)}, \quad (\text{A.19})$$

entailing

$$\frac{\partial \ln J}{\partial y^n} = \frac{1}{N} C_n$$

and

$$\frac{\partial \bar{M}}{\partial y^n} = \frac{1}{h^2} \frac{2}{gNK} A_n - \frac{1}{2} \frac{g}{h^2} \frac{q}{B} \frac{2}{gNK} (2b + gq) A_n + \frac{1}{h^2} \left(-\frac{b}{B} (2b + gq) \frac{2}{gNK} A_n + \frac{2}{gNK} A_n \right),$$

or

$$\frac{\partial \bar{M}}{\partial y^n} = \left(-gq \left(b + \frac{1}{2} gq \right) - b(2b + gq) + 2B \right) \frac{1}{h^2} \frac{1}{B} \frac{2}{gNK} A_n$$

which is equivalent to (A.5).

Starting with (I.3.22)-(I.3.23), we get

$$\begin{aligned} \check{N}^k_{im} &= \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \frac{2}{Ng} A_m l^k + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \frac{1}{q} (b + gq) \mathcal{H}_m^k \\ &- \frac{1}{h^2} g_i \left[\frac{1}{N} \frac{1}{Ng} A_m A^k + l_m \frac{q^2}{B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \frac{1}{Ng} A^k - \frac{2}{q} (b + gq) \frac{2}{Ng} A_m \frac{q^2}{B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \frac{1}{Ng} A^k \right] \\ &- g_i \left(\frac{q^2}{B} \frac{2}{Ng} A_m l^k + \frac{1}{2} \bar{M} \delta_m^k \right) \\ &= \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m l^k + \frac{1}{h^2} g_i \frac{1}{2B} \left(q + \frac{1}{2} gb \right) (b + gq) h_m^k \\ &- \frac{1}{h^2} g_i \frac{1}{B} \left[gB - (2b + 2gq) \left(q + \frac{1}{2} gb \right) \right] \frac{1}{Ng} \frac{1}{Ng} A_m A^k - \frac{1}{2} g_i \bar{M} h_m^k + \frac{1}{K} l_m \check{N}^k_i. \end{aligned}$$

Using here the equality (A.4) leads to

$$\begin{aligned} \check{N}^k_{im} &= \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m l^k + \frac{1}{h^2} g_i \frac{1}{2B} \left(q + \frac{1}{2} gb \right) (b + gq) h_m^k \\ &+ \frac{1}{h^2} g_i \frac{1}{B} \left[2h^2 bq + gq \left(q + \frac{1}{2} gb \right) \right] \frac{1}{Ng} \frac{1}{Ng} A_m A^k - \frac{1}{2} g_i \bar{M} h_m^k + \frac{1}{K} l_m \check{N}^k_i. \end{aligned}$$

Eventually we obtain

$$\begin{aligned}
\check{N}^k_{im} &= \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m l^k + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) h_m^k \\
&\quad + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{2}{Ng} A_m A^k - \frac{1}{2} g_i \bar{M} h_m^k + \frac{1}{K} l_m \check{N}^k_i. \tag{A.20}
\end{aligned}$$

Thus the representation (I.3.28) is valid.

Next, we find that

$$\begin{aligned}
\check{N}^k_{imn} &= \frac{\partial \check{N}^k_{im}}{\partial y^n} = -\frac{1}{h^2} g_i \frac{q}{B} \frac{1}{gNK} (2b + gq) A_n \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m l^k \\
&\quad + \frac{1}{h^2} g_i \frac{1}{2} g \frac{1}{NKg} A_n \frac{2}{Ng} A_m l^k \\
&\quad + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} \frac{1}{K} \left[-A_n l_m + \frac{2}{N} A_n A_m - \frac{b}{q} \frac{Ng}{2} \mathcal{H}_{mn} \right] l^k \\
&\quad + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m \frac{1}{K} h_n^k \\
&\quad - \frac{1}{h^2} g_i \frac{q}{B} \frac{1}{gNK} (2b + gq) A_n \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) h_m^k \\
&\quad + \frac{1}{h^2} g_i \frac{q^2}{2B} \frac{1}{2} g \frac{2B}{NKgq^2} A_n \left(\frac{b}{q} + g \right) h_m^k + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \frac{2B}{NKgq^2} A_n h_m^k \\
&\quad - \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) \frac{1}{K} (l^k h_{mn} + l_m h_n^k) \\
&\quad - \frac{1}{h^2} g_i \frac{q}{B} \frac{1}{gNK} (2b + gq) A_n \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{4}{N^2 g^2} A_m A^k + \frac{1}{h^2} g_i \frac{q^2}{2B} \frac{2B}{NKgq^2} A_n \frac{2}{Ng} \frac{2}{Ng} A_m A^k \\
&\quad - \frac{1}{h^2} g_i \frac{q^2}{2B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{4}{N^2 g^2} \frac{1}{K} \left[A^k \left(A_n l_m + \frac{Ng}{2} \frac{b}{q} \mathcal{H}_{mn} \right) + A_m \left(A_n l^k + \frac{Ng}{2q} (b + gq) \mathcal{H}_n^k \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -g_i \frac{q^2}{B} \frac{2}{gNK} A_n h_m^k + \frac{1}{2} g_i \bar{M} \frac{1}{K} (l^k h_{mn} + l_m h_n^k) \\
& + \frac{1}{K^2} h_{mn} \left[-\frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} g b \right) \frac{K}{Ng} A^k - \frac{1}{2} g_i \bar{M} y^k \right] \\
& + \frac{1}{K} l_m \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_n l^k \\
& + \frac{1}{K} l_m \left[\frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) h_n^k + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{4}{N^2 g^2} A_n A^k - \frac{1}{2} g_i \bar{M} h_n^k \right].
\end{aligned}$$

Simplifying yields

$$\begin{aligned}
\check{N}^k_{imn} = & -\frac{1}{h^2} g_i \frac{bq}{B} A_n \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{4}{N^2 g^2} \frac{1}{K} A_m l^k + \frac{1}{h^2} g_i \frac{1}{2} g \frac{1}{NKg} A_n \frac{2}{Ng} A_m l^k \\
& - \frac{1}{h^2} g_i \frac{bq}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{1}{K} h_{mn} l^k + \frac{1}{h^2} g_i \frac{bq}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{1}{K} \frac{4}{N^2 g^2} A_m A_n l^k \\
& + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m \frac{1}{K} h_n^k \\
& - \frac{1}{h^2} g_i \frac{1}{B} (2b + gq) \left(q + \frac{1}{2} gb \right) \left(\frac{b}{q} + g \right) \frac{1}{gNK} A_n h_m^k \\
& + \frac{1}{h^2} g_i \frac{1}{2} g \frac{1}{NKg} A_n \left(\frac{b}{q} + g \right) h_m^k + \frac{1}{h^2} g_i \left(1 + \frac{1}{2} g \frac{b}{q} \right) \frac{1}{NKg} A_n h_m^k \\
& - \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) \frac{1}{K} l^k h_{mn} \\
& - \frac{1}{h^2} g_i \frac{q}{2B} (2b + gq) \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{2}{Ng} \frac{2}{Ng} \frac{1}{K} A_n A_m A^k \\
& + \frac{1}{h^2} g_i \frac{1}{NKg} A_n \frac{2}{Ng} \frac{2}{Ng} A_m A^k \\
& - \frac{1}{h^2} g_i \frac{q^2}{2B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{2}{Ng} \frac{1}{K} \left[A^k \frac{Ng}{2} \frac{b}{q} \mathcal{H}_{mn} + A_m A_n l^k \right]
\end{aligned}$$

$$-\frac{1}{h^2}g_i\frac{q}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{2}{Ng}\frac{1}{K}(b+gq)A_m\mathcal{H}_n^k$$

$$-g_i\frac{q^2}{B}\frac{2}{gNK}A_nh_m^k-\frac{1}{K^2}h_{mn}\frac{1}{h^2}g_i\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{K}{Ng}A^k,$$

or

$$\check{N}_{imn}^k=\frac{1}{h^2}g_i\frac{1}{2}g\frac{1}{NKg}A_n\frac{2}{Ng}A_m l^k$$

$$-\frac{1}{h^2}g_i\frac{bq}{2B}\left(2+g\frac{b}{q}-2h^2\right)\frac{1}{K}h_{mn}l^k-\frac{1}{h^2}g_i\frac{bq}{2B}\left(1+\frac{1}{2}g\frac{b}{q}-2h^2\right)\frac{1}{K}\frac{2}{Ng}\frac{2}{Ng}A_mA_n l^k$$

$$+\frac{1}{h^2}g_i\frac{q^2}{2B}\left(1+\frac{g}{2}\frac{b}{q}-2h^2\right)\frac{2}{Ng}A_m\frac{1}{K}h_n^k-\frac{1}{h^2}g_i\frac{1}{B}(2h^2bq+gB)\left(\frac{b}{q}+g\right)\frac{1}{gNK}A_nh_m^k$$

$$+\frac{1}{h^2}g_i\frac{1}{2}g\frac{1}{NKg}A_n\left(\frac{b}{q}+g\right)h_m^k+\frac{1}{h^2}g_i\left(1+\frac{1}{2}g\frac{b}{q}\right)\frac{1}{NKg}A_nh_m^k$$

$$-\frac{1}{h^2}g_i\frac{q^2}{2B}\left(1+\frac{1}{2}g\frac{b}{q}\right)g\frac{1}{K}l^kh_{mn}-\frac{1}{h^2}g_i\frac{bq}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{2}{Ng}\frac{2}{Ng}\frac{2}{Ng}\frac{1}{K}A_nA_mA^k$$

$$+\frac{1}{h^2}g_i\frac{1}{NKg}A_n\frac{2}{Ng}\frac{2}{Ng}A_mA^k$$

$$-\frac{1}{h^2}g_i\frac{q^2}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{2}{Ng}\frac{1}{K}A^k\frac{b}{q}\mathcal{H}_{mn}-\frac{1}{h^2}g_i\frac{q^2}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{2}{Ng}\frac{2}{Ng}\frac{1}{K}A_mA_n l^k$$

$$-\frac{1}{h^2}g_i\frac{q}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{2}{Ng}\frac{1}{K}(b+gq)A_m h_n^k$$

$$-g_i\frac{q^2}{B}\frac{2}{gNK}A_nh_m^k-\frac{1}{K^2}h_{mn}\frac{1}{h^2}g_i\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{K}{Ng}A^k.$$

Additional reductions are possible, leading to

$$\begin{aligned}
\check{N}^k_{imn} &= \frac{1}{h^2} g_i \frac{1}{2} g \frac{1}{NKg} A_n \frac{2}{Ng} A_m l^k \\
&- \frac{g}{2h^2} g_i \frac{1}{K} h_{mn} l^k - \frac{1}{h^2} g_i \frac{bq}{2B} \left(2 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{1}{K} \frac{2}{Ng} \frac{2}{Ng} A_m A_n l^k \\
&+ \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m \frac{1}{K} h_n^k \\
&- \frac{1}{h^2} g_i \left(2h^2 + g \left(\frac{b}{q} + g \right) \right) \frac{1}{gNK} A_n h_m^k \\
&+ \frac{1}{h^2} g_i \frac{1}{2} g \frac{1}{NKg} A_n \left(\frac{b}{q} + g \right) h_m^k + \frac{1}{h^2} g_i \left(1 + \frac{1}{2} g \frac{b}{q} \right) \frac{1}{NKg} A_n h_m^k \\
&+ \frac{1}{h^2} g_i \frac{1}{NKg} A_n \frac{2}{Ng} \frac{2}{Ng} A_m A^k \\
&- \frac{1}{h^2} g_i \frac{bq}{2B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{1}{K} A^k h_{mn} - \frac{1}{h^2} g_i \frac{q^2}{2B} \frac{1}{2} g \frac{2}{Ng} \frac{2}{Ng} \frac{1}{K} A_m A_n l^k \\
&- \frac{1}{h^2} g_i \frac{q}{2B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{1}{K} (b + gq) A_m h_n^k - \frac{1}{K^2} h_{mn} \frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \frac{K}{Ng} A^k,
\end{aligned}$$

or simply

$$\begin{aligned}
\check{N}^k_{imn} &= -\frac{g}{2h^2} g_i \frac{1}{K} h_{mn} l^k + \frac{1}{h^2} g_i \frac{1}{2B} \left(-q^2 + \frac{1}{2} gbq + \frac{1}{2} g^2 q^2 \right) \frac{2}{Ng} \frac{1}{K} h_n^k A_m \\
&- \frac{1}{h^2} g_i \left(2 + \frac{g^2}{2} + g \frac{b}{q} \right) \frac{1}{gNK} A_n h_m^k + \frac{1}{h^2} g_i \frac{g}{2NK} A_n h_m^k + \frac{1}{h^2} g_i \left(1 + g \frac{b}{q} \right) \frac{1}{NKg} A_n h_m^k \\
&+ \frac{1}{h^2} g_i \frac{1}{NKg} A_n \frac{2}{Ng} \frac{2}{Ng} A_m A^k \\
&- \frac{1}{h^2} g_i \frac{1}{2B} \left(b + \frac{1}{2} gq \right) (b + gq) \frac{2}{Ng} \frac{1}{K} A_m h_n^k - \frac{1}{h^2} g_i \frac{1}{Ng} \frac{1}{K} h_{mn} A^k. \tag{A.21}
\end{aligned}$$

Using here the representation (A.14) of the tensor A_{ijk} , we are coming to

$$y_k \frac{\partial^2 \check{N}^k_i}{\partial y^m \partial y^n} = \frac{2}{h} h_i h_{mn}. \quad (\text{A.22})$$

Let us verify the validity of the equality

$$\mathcal{D}_i h_{nm} = -\frac{2}{h} h_i h_{nm}. \quad (\text{A.23})$$

To this end we find

$$\begin{aligned} 2\check{N}^k_i C_{kmn} &= -\frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \frac{2}{Ng} A^k A_{kmn} \\ &= -\frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \frac{2}{Ng} A^k \frac{1}{N} \left[A_n h_{nk} + A_n h_{mk} + A_k h_{mn} - \frac{4}{N^2 g^2} A_m A_n A_k \right], \end{aligned}$$

so that

$$2\check{N}^k_i C_{kmn} = -\frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \frac{2}{Ng} \frac{1}{N} \left[A_m A_n + \frac{N^2 g^2}{4} h_{mn} \right]. \quad (\text{A.24})$$

We can also observe that

$$\begin{aligned} g_i \frac{\partial g_{mn}}{\partial g} + 2\check{N}^k_i C_{kmn} &= g_i \bar{M} g_{mn} + g_i \frac{q^2}{B} \frac{2}{gN} (A_m l_n + A_n l_m) - g_i \frac{bq}{B} \frac{4}{N^2 g^2} A_m A_n - g_i \frac{bq}{B} h_{mn} \\ &\quad - \frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \left[\frac{2}{Ng} \frac{1}{N} A_m A_n + \frac{g}{2} h_{mn} \right]. \end{aligned}$$

Simultaneously,

$$\begin{aligned} g_{kn} \check{N}^k_{im} &= \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} A_m l_n + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) h_{nm} \\ &\quad + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{2}{Ng} A_m A_n - \frac{1}{2} g_i \bar{M} h_{mn} \\ &\quad + \frac{1}{K} l_m \left[-\frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \frac{K}{Ng} A_n - \frac{1}{2} g_i \bar{M} y_n \right]. \end{aligned}$$

In this way we obtain

$$g_i \frac{\partial g_{mn}}{\partial g} + 2\check{N}^k_i C_{kmn} + g_{km} \check{N}^k_{in} + g_{kn} \check{N}^k_{im}$$

$$\begin{aligned}
&= g_i \frac{q^2}{B} \frac{2}{gN} (A_m l_n + A_n l_m) - g_i \frac{bq}{B} \frac{2}{gN} \frac{2}{gN} A_m A_n - g_i \frac{bq}{B} h_{mn} \\
&\quad - \frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \left[\frac{2}{Ng} \frac{1}{N} A_m A_n + \frac{g}{2} h_{mn} \right] \\
&\quad + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2 \right) \frac{2}{Ng} (A_m l_n + A_n l_m) + \frac{1}{h^2} g_i \frac{q^2}{B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) h_{nm} \\
&\quad + \frac{1}{h^2} g_i \frac{q^2}{B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{2}{Ng} A_m A_n - \frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \frac{1}{Ng} (l_m A_n + l_n A_m).
\end{aligned}$$

Reducing similar terms leads to

$$\begin{aligned}
&g_i \frac{\partial g_{mn}}{\partial g} + 2\check{N}^k{}_i C_{kmn} + g_{km} \check{N}^k{}_{in} + g_{kn} \check{N}^k{}_{im} \\
&= -g_i \frac{bq}{B} \frac{2}{gN} \frac{2}{gN} A_m A_n - g_i \frac{bq}{B} h_{mn} - \frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \left[\frac{2}{Ng} \frac{1}{N} A_m A_n + \frac{g}{2} h_{mn} \right] \\
&\quad + \frac{1}{h^2} g_i \frac{q^2}{B} \left(1 + \frac{1}{2} g \frac{b}{q} \right) \left(\frac{b}{q} + g \right) h_{nm} + \frac{1}{h^2} g_i \frac{q^2}{B} \left(\frac{b}{q} + \frac{1}{2} g \right) \frac{2}{Ng} \frac{2}{Ng} A_m A_n \\
&= -g_i \frac{bq}{B} h_{mn} + \frac{1}{h^2} g_i \frac{1}{2B} (2h^2 bq + gB) h_{nm} = \frac{1}{2h^2} g g_i h_{nm}.
\end{aligned}$$

We get

$$\mathcal{D}_i g_{nm} = \frac{1}{2h^2} g g_i h_{nm}. \quad (\text{A.25})$$

Thus the equality (A.23) is valid.

Now we want to verify the validity of the equality (A.9). Differentiating (A.14) with respect to y^j yields

$$\begin{aligned}
2 \frac{\partial C_{mnj}}{\partial g} &= 2\bar{M} C_{mnj} + 2 \frac{q^2}{B} \frac{2}{gNK} A_j g_{mn} \\
&\quad - \frac{q}{B} \frac{2}{gNK} (2b + gq) A_j \frac{2}{gN} (A_m l_n + A_n l_m) + \frac{1}{K} \frac{q^2}{B} \frac{2}{gN} (A_m h_{nj} + A_n h_{mj})
\end{aligned}$$

$$\begin{aligned}
& + \frac{q^2}{B} \frac{2}{gN} \frac{1}{K} \left[\left(-A_j l_m + \frac{2}{N} A_m A_j - \frac{b}{q} \frac{Ng}{2} \mathcal{H}_{mj} \right) l_n + \left(-A_j l_n + \frac{2}{N} A_n A_j - \frac{b}{q} \frac{Ng}{2} \mathcal{H}_{nj} \right) l_m \right] \\
& + \left(\frac{b}{B} (2b + gq) \frac{2}{gNK} A_j - \frac{2}{gNK} A_j \right) \frac{2}{gN} \frac{2}{gN} A_m A_n \\
& - \frac{bq}{B} \frac{4}{g^2 N^2} \frac{1}{K} \left[\left(-A_j l_m + \frac{2}{N} A_m A_j - \frac{b}{q} \frac{Ng}{2} \mathcal{H}_{mj} \right) A_n + \left(-A_j l_n + \frac{2}{N} A_n A_j - \frac{b}{q} \frac{Ng}{2} \mathcal{H}_{nj} \right) A_m \right] \\
& + \left(\frac{b}{B} (2b + gq) \frac{2}{gNK} A_j - \frac{2}{gNK} A_j \right) h_{mn} + \frac{bq}{B} \frac{1}{K} (l_m h_{jn} + l_n h_{mj}) - \frac{2bq}{B} C_{mnj}.
\end{aligned}$$

We may reduce as follows:

$$\begin{aligned}
2 \frac{\partial C_{mnj}}{\partial g} &= 2 \bar{M} C_{mnj} + 2 \frac{q^2}{B} \frac{2}{gNK} A_j h_{mn} + \frac{1}{K} \frac{q^2}{B} \frac{2}{gN} (A_m h_{nj} + A_n h_{mj}) \\
& + \left(\frac{B - q^2 + b^2}{B} \frac{2}{gNK} - \frac{2}{gNK} \right) \frac{2}{gN} \frac{2}{gN} A_j A_m A_n \\
& - \frac{bq}{B} \frac{2}{gN} \frac{2}{gN} \frac{1}{K} \left[\left(\frac{2}{N} A_m A_j - \frac{b}{q} \frac{Ng}{2} h_{mj} + \frac{b}{q} \frac{2}{Ng} A_m A_j \right) A_n \right. \\
& \quad \left. + \left(\frac{2}{N} A_n A_j - \frac{b}{q} \frac{Ng}{2} h_{nj} + \frac{b}{q} \frac{2}{Ng} A_n A_j \right) A_m \right] \\
& + \left(\frac{B - q^2 + b^2}{B} \frac{2}{gNK} - \frac{2}{gNK} \right) A_j h_{mn} - \frac{2bq}{B} C_{mnj} \\
& = 2 \bar{M} C_{mnj} + \frac{1}{K} \frac{B - gbq}{B} \frac{2}{gN} (A_m h_{nj} + A_n h_{mj} + A_j h_{mn}) - \frac{2bq}{B} C_{mnj} \\
& + \frac{b^2 - q^2}{B} \frac{2}{gNK} \frac{2}{gN} \frac{2}{gN} A_j A_m A_n - \frac{2bq}{B} \frac{2}{gN} \frac{2}{gN} \frac{1}{K} \left(\frac{2}{N} A_m A_j + \frac{b}{q} \frac{2}{Ng} A_m A_j \right) A_n \\
& = 2 \bar{M} \frac{1}{K} A_{mnj} + \frac{1}{K} \frac{B - gbq}{B} \frac{2}{gN} (A_m h_{nj} + A_n h_{mj} + A_j h_{mn})
\end{aligned}$$

$$\begin{aligned}
& -\frac{B+gbq}{B} \frac{2}{gNK} \frac{2}{gN} \frac{2}{gN} A_j A_m A_n - \frac{2bq}{B} C_{mnj} \\
& = 2\bar{M} \frac{1}{K} A_{mnj} + \frac{2}{K} \left(\frac{1}{g} - \frac{bq}{B} \right) A_{mnj} - \frac{1}{K} \frac{gbq}{B} \frac{2}{gN} (A_m h_{nj} + A_n h_{mj} + A_j h_{mn}) \\
& \quad - \frac{gbq}{B} \frac{2}{gNK} \frac{2}{gN} \frac{2}{gN} A_j A_m A_n.
\end{aligned}$$

Thus (A.9) is valid.

Next, we evaluate the term

$$\check{N}^k_i \frac{\partial A_{jmn}}{\partial y^k} = -\frac{1}{h^2} g_i \frac{q}{B} \left(q + \frac{1}{2} gb \right) \frac{K}{Ng} A^k \frac{\partial A_{jmn}}{\partial y^k}.$$

With the representation

$$\begin{aligned}
\frac{\partial A_{ijk}}{\partial y^n} &= \frac{1}{K} \frac{2}{N} (A_{jkn} A_i + A_{ikn} A_j + A_{ijn} A_k) - \frac{1}{K} (l_j A_{kni} + l_i A_{knj} + l_k A_{ijn}) \\
&+ \frac{1}{K} \frac{1}{N} \frac{2}{N} (\mathcal{H}_{jk} A_i A_n + \mathcal{H}_{ik} A_j A_n + \mathcal{H}_{ij} A_k A_n) - \frac{gb}{2Kq} (\mathcal{H}_{jk} \mathcal{H}_{in} + \mathcal{H}_{ik} \mathcal{H}_{jn} + \mathcal{H}_{ji} \mathcal{H}_{kn})
\end{aligned} \tag{A.26}$$

we obtain

$$\begin{aligned}
KA^n \frac{\partial A_{ijk}}{\partial y^n} &= \frac{2}{N} (A_{jkn} A_i + A_{ikn} A_j + A_{ijn} A_k) A^n - (l_j A_{kni} + l_i A_{knj} + l_k A_{ijn}) A^n \\
&+ \frac{g^2}{2} (\mathcal{H}_{jk} A_i + \mathcal{H}_{ik} A_j + \mathcal{H}_{ij} A_k).
\end{aligned}$$

Using here the equality

$$A_{jkn} A^n = \frac{1}{N} \left(\frac{Ng}{2} \frac{Ng}{2} h_{jk} + A_j A_k \right)$$

leads to

$$\begin{aligned}
KA^n \frac{\partial A_{ijk}}{\partial y^n} &= \frac{2}{N} \frac{1}{N} \left(\frac{Ng}{2} \frac{Ng}{2} (h_{jk} A_i + h_{ik} A_j + h_{ij} A_k) + 3A_i A_j A_k \right) \\
&- \frac{1}{N} \frac{Ng}{2} \frac{Ng}{2} (h_{jk} l_i + h_{ik} l_j + h_{ij} l_k) - \frac{1}{N} (A_j A_k l_i + A_i A_k l_j + A_i A_j l_k)
\end{aligned}$$

$$+\frac{g^2}{2}(h_{jk}A_i+h_{ik}A_j+h_{ij}A_k)-3\frac{g^2}{2}\frac{2}{Ng}\frac{2}{Ng}A_iA_jA_k.$$

So we may write

$$\begin{aligned} KA^n\frac{\partial A_{ijk}}{\partial y^n} &= -\frac{1}{N}\frac{Ng}{2}\frac{Ng}{2}(h_{jk}l_i+h_{ik}l_j+h_{ij}l_k)-\frac{1}{N}(A_jA_kl_i+A_iA_kl_j+A_iA_jl_k) \\ &\quad +g^2(h_{jk}A_i+h_{ik}A_j+h_{ij}A_k). \end{aligned} \quad (\text{A.27})$$

We also need the term

$$\begin{aligned} A_{jkn}\check{N}^k_{im} &= \frac{1}{h^2}g_i\frac{q^2}{2B}\left(1+\frac{1}{2}g\frac{b}{q}\right)\left(\frac{b}{q}+g\right)A_{jmn} \\ &\quad +\frac{1}{h^2}g_i\frac{q^2}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{2}{Ng}\frac{2}{Ng}A_m\frac{1}{N}\left(\frac{Ng}{2}\frac{Ng}{2}h_{jn}+A_jA_n\right)-\frac{1}{2}g_i\bar{M}A_{jmn} \\ &\quad -l_m\frac{1}{h^2}g_i\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{1}{Ng}\frac{1}{N}\left(A_jA_n+\frac{N^2g^2}{4}h_{jn}\right). \end{aligned}$$

Summing all the addends yields

$$\begin{aligned} &g_i\frac{\partial A_{jmn}}{\partial g}+\check{N}^k_i\frac{\partial A_{jmn}}{\partial y^k}+A_{jkm}\check{N}^k_{in}+A_{jkn}\check{N}^k_{im}+A_{mnk}\check{N}^k_{ij} \\ &= g_i\left(\frac{1}{g}-2\frac{bq}{B}\right)A_{mnj}-g_i\frac{2bq}{B}\frac{1}{N}\frac{2}{gN}\frac{2}{gN}A_jA_mA_n \\ &\quad -\frac{1}{h^2}g_i\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{1}{Ng}\left[-\frac{Ng^2}{4}(h_{jm}l_n+h_{mn}l_j+h_{nj}l_m)-\frac{1}{N}(A_jA_ml_n+A_mA_jl_n+A_nA_jl_m)\right. \\ &\quad \left.+g^2(h_{mn}A_j+h_{mj}A_n+h_{nj}A_m)\right] \\ &\quad +3\frac{1}{h^2}g_i\frac{1}{2B}\left(q+\frac{1}{2}gb\right)(b+gq)A_{jmn} \\ &\quad +\frac{1}{h^2}g_i\frac{q^2}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{1}{N}(h_{mn}A_j+h_{mj}A_n+h_{nj}A_m)+\frac{3}{h^2}g_i\frac{q^2}{2B}\left(\frac{b}{q}+\frac{1}{2}g\right)\frac{4}{N^3g^2}A_mA_jA_n \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{h^2}g_i\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{1}{Ng}\frac{1}{N}(A_jA_m l_n + A_m A_j l_n + A_n A_j l_m) \\
& -\frac{1}{h^2}g_i\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{g}{4}(h_{jm}l_n + h_{mn}l_j + h_{nj}l_m),
\end{aligned}$$

or

$$\begin{aligned}
& g_i\frac{\partial A_{jmn}}{\partial g} + \check{N}^k{}_i\frac{\partial A_{jmn}}{\partial y^k} + A_{jkm}\check{N}^k{}_{in} + A_{jkn}\check{N}^k{}_{im} + A_{mnk}\check{N}^k{}_{ij} \\
& = g_i\left(\frac{1}{g} - 2\frac{bq}{B}\right)A_{mnj} - g_i\frac{2bq}{B}\frac{1}{N}\frac{2}{gN}\frac{2}{gN}A_jA_mA_n \\
& -\frac{1}{h^2}g_i\frac{g^2q}{B}\left(q+\frac{1}{2}gb\right)\frac{1}{Ng}(h_{mn}A_j + h_{mj}A_n + h_{nj}A_m) \\
& +\frac{1}{h^2}g_i\frac{1}{2B}\left[6h^2bq + 3gB - b\left(3q + \frac{3}{2}gb\right)\right]A_{jmn} \\
& +\frac{1}{h^2}g_i\frac{gq}{2B}\left(b + \frac{g}{2}q\right)\frac{1}{Ng}(h_{mn}A_j + h_{mj}A_n + h_{nj}A_m) + \frac{3}{h^2}g_i\frac{gq}{2B}\left(b + \frac{g}{2}q\right)\frac{4}{N^3g^3}A_mA_jA_n \\
& = g_i\frac{1}{g}A_{jmn} + \frac{1}{h^2}g_i\frac{1}{2B}\left[4h^2bq + 3gB - b\left(3q + \frac{3}{2}gb\right)\right]A_{jmn} \\
& +\frac{1}{h^2}g_i\frac{q}{2B}\left(b - 2h^2b - \frac{3}{2}gq - g^2b\right)A_{jmn} = g_i\frac{1}{g}A_{jmn} + \frac{3}{4h^2}g_iA_{jmn}.
\end{aligned}$$

By the help of such evaluations we eventually obtain

$$g_i\frac{\partial A_{jmn}}{\partial g} + \check{N}^k{}_i\frac{\partial A_{jmn}}{\partial y^k} + A_{jkm}\check{N}^k{}_{in} + A_{jkn}\check{N}^k{}_{im} + A_{mnk}\check{N}^k{}_{ij} = g_i\frac{1}{gh^2}A_{jmn} + \frac{1}{h^2}g_iA_{jmn}, \quad (\text{A.28})$$

which shows that the representation

$$\check{N}^k{}_{imn} = -\frac{g}{2h^2}g_i\frac{1}{K}h_{mn}l^k - \frac{1}{gh^2}g_i\frac{1}{K}A^k{}_{mn} \quad (\text{A.29})$$

indicated in (I.3.30) is valid.

The full coefficients $N^k_{imn} = N^{Ik}_{imn} + \check{N}^k_{imn}$ can be obtained on taking into account the components (A.29) together with the representation $N^{Ik}_{imn} = -(1/K)\mathcal{D}_i A^k_{mn}$ obtainable in the $(g = \text{const})$ -case (see [10,11]). The result reads

$$N^k_{imn} = \frac{2}{h} h_i \frac{1}{K} l^k h_{mn} - \frac{1}{K} \mathcal{D}_i A^k_{mn}. \quad (\text{A.30})$$

Thus the representation indicated in (I.3.31) is also valid.

Appendix B: Conformal property of the tangent Riemannian space

Given an arbitrary Finsler space of any dimension $N \geq 3$. At any fixed point x , the Riemannian curvature tensor $\widehat{R}_{\{x\}} = \{\widehat{R}_n{}^m{}_{ij}(x, y)\}$ of the tangent Riemannian space $\mathcal{R}_{\{x\}}$ is given by means of the components

$$\widehat{R}_n{}^m{}_{ij} = \frac{1}{F^2} S_n{}^m{}_{ij}, \quad (\text{B.1})$$

where

$$S_n{}^m{}_{ij} = (C^h{}_{nj} C^m{}_{hi} - C^h{}_{ni} C^m{}_{hj}) F^2. \quad (\text{B.2})$$

Let us construct the Weyl tensor W_{ijmn} in the space $\mathcal{R}_{\{x\}}$, so that

$$F^2 W_{ijmn} = S_{ijmn}$$

$$- \frac{1}{N-2} (S_{im} g_{jn} + S_{jn} g_{im} - S_{in} g_{jm} - S_{jm} g_{in}) + \frac{1}{(N-1)(N-2)} \check{S} (g_{im} g_{jn} - g_{in} g_{jm}), \quad (\text{B.3})$$

where $S_{ijmn} = g_{jh} S_i{}^j{}_{mn}$, $S_{im} = g^{jn} S_{ijmn}$ and $\check{S} = g^{im} S_{im}$. Contracting the tensor two times by the unit vector $l^n = (1/F)y^n$ yields directly

$$(N-2) F^2 W_{ijmn} l^n l^j = -S_{im} + \frac{1}{N-1} \check{S} h_{im},$$

where $h_{im} = g_{im} - (1/F^2) y_i y_m$. Therefore, in any dimension $N \geq 4$ the vanishing $W_{ijmn} = 0$ is tantamount to the representation

$$S_{nmij} = C(h_{nj} h_{mi} - h_{ni} h_{mj}). \quad (\text{B.4})$$

It is known (see Section 5.8 in [1]) that the indicatrix is a space of constant curvature if and only if the tensor (B.2) fulfills the representation (B.4), in which case $C = C(x)$ (that is, the factor C is independent of y). The respective indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ is given by

$$\mathcal{C}_{\text{Ind.}} = 1 - C. \quad (\text{B.5})$$

Next, in the dimension $N = 3$ the tensor W_{ijmn} vanishes identically and, therefore, the equality

$$S_{ijmn} = L(h_{im} h_{jn} - h_{in} h_{jm}) \quad \text{with} \quad L = \frac{1}{2} \check{S} \quad (\text{B.6})$$

holds, where L may depend on y . Taking $S_{im} = Lh_{im}$, we should examine the tensor

$$C_{im} := \frac{1}{F^2} \left(S_{im} - \frac{1}{4} \check{S} g_{im} \right) \quad (\text{B.7})$$

of the Cotton-York type. Let us use the Riemannian covariant derivative \mathcal{S} operative in the space $\mathcal{R}_{\{x\}}$ under consideration. Denoting $L_n = \partial L / \partial y^n$ and taking into account the vanishing $\mathcal{S}_n g_{im} = 0$, we have

$$\begin{aligned} \mathcal{S}_n C_{im} - \mathcal{S}_m C_{in} &= \frac{1}{F^2} \left(L_n h_{im} - L_m h_{in} - L \frac{1}{F} (l_n h_{mi} - l_m h_{ni}) - \frac{1}{2} (L_n g_{im} - L_m g_{in}) \right) \\ &\quad + L \frac{1}{F^3} (l_n h_{im} - l_m h_{in}), \end{aligned}$$

which is

$$\mathcal{S}_n C_{im} - \mathcal{S}_m C_{in} = \frac{1}{F^2} \left(L_n \left(h_{im} - \frac{1}{2} g_{im} \right) - L_m \left(h_{in} - \frac{1}{2} g_{in} \right) \right),$$

so that

$$\mathcal{S}_n C_{im} - \mathcal{S}_m C_{in} = 0 \quad (\text{B.8})$$

holds iff $L_n = 0$, that is when $\check{S} = \check{S}(x)$. The vanishing (B.8) means the conformal flatness of the three-dimensional space $\mathcal{R}_{\{x\}}$.

Thus we are entitled to set forth the validity of the following proposition.

Proposition. *Given an arbitrary Finsler space of any dimension $N \geq 3$. The tangent Riemannian space $\mathcal{R}_{\{x\}}$ is conformally flat if and only if the indicatrix is a space of constant curvature.*

The question arises: What is the form of the conformal multiplier of the space $\mathcal{R}_{\{x\}}$ under study? See the next appendix.

Appendix C: Multiplier for the tangent Riemannian space

To find the form of the conformal multiplier of the space $\mathcal{R}_{\{x\}}$ under study, we can start with the conformal tensor

$$u_{ij} = \frac{z(x, y)}{(c_1(x))^2 F^{2a}(x)} g_{ij} \quad (\text{C.1})$$

(cf. (II.2.3)), where z is a test smooth positive function.

Denoting $u_{ijk} = \partial u_{ij} / \partial y^k$, we get

$$(c_1)^2 u_{ijk} = \frac{1}{F^{2a}} \left(-2z \frac{a}{F} l_k + z_k \right) g_{ij} + 2z \frac{1}{F^{2a}} C_{ijk},$$

where $C_{ijk} = (1/2)\partial g_{ij}/\partial y^k$ and $z_k = \partial z/\partial y^k$.

Constructing the coefficients

$$Z_{ijk} := \frac{1}{2}(u_{kji} + u_{iki} - u_{ijk})$$

leads to

$$F^{2a}(c_1)^2 Z_{ijk} = \left(-\frac{za}{F}l_i + \frac{1}{2}z_i\right)g_{kj} + \left(-\frac{za}{F}l_j + \frac{1}{2}z_j\right)g_{ik} - \left(-\frac{za}{F}l_k + \frac{1}{2}z_k\right)g_{ij} + zC_{ijk}.$$

Since the components u^{ij} reciprocal to the components (C.1) are of the form $u^{ij} = (1/z)F^{2a}g^{ij}(c_1)^2$, the coefficients $Z^m_{ij} = u^{mh}Z_{ijh}$ read merely

$$Z^m_{ij} = \left(-\frac{a}{F}l_i + \frac{1}{2z}z_i\right)\delta_j^m + \left(-\frac{a}{F}l_j + \frac{1}{2z}z_j\right)\delta_i^m - \left(-\frac{a}{F}l^m + \frac{1}{2z}g^{mk}z_k\right)g_{ij} + C^m_{ij}. \quad (\text{C.2})$$

We straightforwardly obtain

$$\begin{aligned} \frac{\partial Z^m_{ni}}{\partial y^j} &= \frac{a}{F^2}l_j(l_n\delta_i^m + l_i\delta_n^m - l^m g_{ni}) \\ &- \frac{a}{F^2} \left[h_{ij}\delta_n^m + h_{nj}\delta_i^m - h_j^m g_{in} - 2 \left(l^m - \frac{1}{2z}\frac{F}{a}g^{mk}z_k \right) FC_{inj} \right] + \frac{\partial C^m_{ni}}{\partial y^j} \\ &+ \frac{1}{2} \left[(\ln z)_{nj}\delta_i^m + (\ln z)_{ij}\delta_n^m - \frac{\partial(g^{mk}(\ln z)_k)}{\partial y^j}g_{ni} \right] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial Z^m_{ni}}{\partial y^j} - \frac{\partial Z^m_{nj}}{\partial y^i} &= \frac{a}{F^2}[l_n(l_j\delta_i^m - l_i\delta_j^m) - l^m(l_jg_{ni} - l_i g_{nj})] \\ &- \frac{a}{F^2}[(h_{nj}\delta_i^m - h_{ni}\delta_j^m) - (h_j^m g_{in} - h_i^m g_{jn})] + \frac{\partial C^m_{ni}}{\partial y^j} - \frac{\partial C^m_{nj}}{\partial y^i} \\ &+ \frac{1}{2} \left((\ln z)_{nj}\delta_i^m - (\ln z)_{ni}\delta_j^m \right) - \frac{1}{2} \left(\frac{\partial(g^{mk}(\ln z)_k)}{\partial y^j}g_{ni} - \frac{\partial(g^{mk}(\ln z)_k)}{\partial y^i}g_{nj} \right), \end{aligned}$$

so that

$$\frac{\partial Z^m_{ni}}{\partial y^j} - \frac{\partial Z^m_{nj}}{\partial y^i} = -\frac{2a}{F^2}(h_{nj}h_i^m - h_{ni}h_j^m) + \frac{\partial C^m_{ni}}{\partial y^j} - \frac{\partial C^m_{nj}}{\partial y^i}$$

$$+\frac{1}{2}\left((\ln z)_{nj}\delta_i^m-(\ln z)_{ni}\delta_j^m\right)-\frac{1}{2}g^{mh}\left[(\ln z)_{hj}g_{ni}-(\ln z)_{hi}g_{nj}\right]+(\ln z)_h\left[C^{mh}_{j}g_{ni}-C^{mh}_{i}g_{nj}\right]. \quad (\text{C.3})$$

Also,

$$Z^h_{ni}Z^m_{hj}-Z^h_{nj}Z^m_{hi}=-\frac{a}{F}\left[-\frac{a}{F}\left[l_n(l_i\delta_j^m-l_j\delta_i^m)+l^m(l_jg_{ni}-l_i g_{nj})\right]+(l_iC^m_{nj}-l_jC^m_{ni})\right]$$

$$-\frac{a}{F}l_i\delta_n^h\left[\frac{1}{2z}z_h\delta_j^m-\frac{1}{2z}g^{mk}z_kg_{hj}\right]-[ij]$$

$$+\frac{1}{2z}z_i\delta_n^h\left[\left(-\frac{a}{F}l_h+\frac{1}{2z}z_h\right)\delta_j^m-\left(-\frac{a}{F}l^m+\frac{1}{2z}g^{mk}z_k\right)g_{hj}+C^m_{hj}\right]-[ij]$$

$$-\left(\frac{a}{F}\right)^2(g_{in}\delta_j^m-g_{jn}\delta_i^m)$$

$$+\frac{a}{F}l^h g_{in}\left[\frac{1}{2z}z_h\delta_j^m+\frac{1}{2z}z_j\delta_h^m-\frac{1}{2z}g^{mk}z_kg_{hj}\right]-[ij]$$

$$-\frac{1}{2z}g^{hk}z_kg_{in}\left[\left(-\frac{a}{F}l_h+\frac{1}{2z}z_h\right)\delta_j^m+\left(-\frac{a}{F}l_j+\frac{1}{2z}z_j\right)\delta_h^m-\left(-\frac{a}{F}l^m+\frac{1}{2z}g^{mk}z_k\right)g_{hj}\right]-[ij]$$

$$-\frac{1}{2z}g^{hk}z_k(g_{in}C^m_{hj}-g_{jn}C^m_{hi})$$

$$-\frac{a}{F}l_jC^m_{in}+C^h_{in}\left[\frac{1}{2z}z_h\delta_j^m+\frac{1}{2z}z_j\delta_h^m-\frac{1}{2z}g^{mk}z_kg_{hj}\right]-[ij]$$

$$+C^h_{ni}C^m_{hj}-C^h_{nj}C^m_{hi},$$

or

$$Z^h_{ni}Z^m_{hj}-Z^h_{nj}Z^m_{hi}=-\left(\frac{a}{F}\right)^2(h_{in}h_j^m-h_{jn}h_i^m)+C^h_{ni}C^m_{hj}-C^h_{nj}C^m_{hi}$$

$$\begin{aligned}
& -\frac{a}{F}l_i \left[\frac{1}{2z}z_n\delta_j^m - \frac{1}{2z}g^{mk}z_k g_{nj} \right] + \frac{1}{2z}z_i \left(-\frac{a}{F}l_n + \frac{1}{2z}z_n \right) \delta_j^m - [ij] \\
& + \frac{a}{F}l^h g_{in} \left[\frac{1}{2z}z_h\delta_j^m - \frac{1}{2z}g^{mk}z_k g_{hj} \right] - [ij] \\
& - \frac{1}{2z}g^{hs}z_s g_{in} \left[\left(-\frac{a}{F}l_h + \frac{1}{2z}z_h \right) \delta_j^m - \frac{a}{F}l_j\delta_h^m + \frac{a}{F}l^m g_{hj} - \frac{1}{2z}g^{mk}z_k g_{hj} \right] - [ij] \\
& + \frac{1}{2z}z_h \left[-C^{hm}{}_j g_{in} + C^{hm}{}_i g_{jn} + C^h{}_{in}\delta_j^m - C^h{}_{jn}\delta_i^m \right].
\end{aligned}$$

In this way we come to

$$\begin{aligned}
Z^h{}_{ni}Z^m{}_{hj} - Z^h{}_{nj}Z^m{}_{hi} &= -\left(\frac{a}{F}\right)^2 (h_{in}h_j^m - h_{jn}h_i^m) + C^h{}_{ni}C^m{}_{hj} - C^h{}_{nj}C^m{}_{hi} \\
& - \frac{a}{F}\frac{1}{2z}z_n(l_i\delta_j^m - l_j\delta_i^m) + \frac{a}{F}\frac{1}{2z}g^{mk}z_k(l_i g_{nj} - l_j g_{ni}) \\
& + \frac{1}{2z} \left(-\frac{a}{F}l_n + \frac{1}{2z}z_n \right) (z_i\delta_j^m - z_j\delta_i^m) \\
& + \frac{a}{F}\frac{1}{z}(l^s z_s)(g_{in}\delta_j^m - g_{jn}\delta_i^m) - \frac{1}{4z^2}(z_h g^{hs}z_s)(g_{in}\delta_j^m - g_{jn}\delta_i^m) \\
& + \frac{1}{2z} \left(-\frac{a}{F}l^m + \frac{1}{2z}g^{mk}z_k \right) (z_j g_{in} - z_i g_{jn}) \\
& + \frac{1}{2z}z_h \left[-C^{hm}{}_j g_{in} + C^{hm}{}_i g_{jn} + C^h{}_{in}\delta_j^m - C^h{}_{jn}\delta_i^m \right]. \tag{C.4}
\end{aligned}$$

Thus we are able to evaluate the curvature tensor

$$\tilde{R}_n{}^m{}_{ij} := \frac{\partial Z^m{}_{ni}}{\partial y^j} - \frac{\partial Z^m{}_{nj}}{\partial y^i} + Z^h{}_{ni}Z^m{}_{hj} - Z^h{}_{nj}Z^m{}_{hi}.$$

By lowering the index

$$\tilde{R}_{nmij} := u_{mt}\tilde{R}_n{}^t{}_{ij},$$

we obtain the representation

$$\begin{aligned}
\frac{1}{z}(c_1(x))^2 F^{2a(x)} \tilde{R}_{nmij} &= \frac{1}{F^2}(a^2 - 2a)(h_{nj}h_{mi} - h_{ni}h_{mj}) + \frac{1}{F^2}S_{nmij} \\
&+ \frac{1}{2} \left((\ln z)_{nj}g_{mi} - (\ln z)_{ni}g_{mj} - (\ln z)_{mj}g_{ni} + (\ln z)_{mi}g_{nj} \right) \\
&- \frac{a}{F} \frac{1}{2z} \left(z_n(l_i h_{mj} - l_j h_{mi}) - z_m(l_i h_{nj} - l_j h_{ni}) \right) \\
&+ \frac{1}{2z} \left(-\frac{a}{F}l_n + \frac{1}{2z}z_n \right) (z_i g_{mj} - z_j g_{mi}) - \frac{1}{2z} \left(-\frac{a}{F}l_m + \frac{1}{2z}z_m \right) (z_i g_{jn} - z_j g_{in}) \\
&+ \frac{a}{F} \frac{1}{z} (l^s z_s) (g_{in}g_{mj} - g_{jn}g_{mi}) - \frac{1}{4z^2} (z_h g^{hs} z_s) (g_{in}g_{mj} - g_{jn}g_{mi}) \\
&+ \frac{1}{2z} z_h \left(C^h_{mj}g_{in} - C^h_{mi}g_{jn} - C^h_{nj}g_{mi} + C^h_{in}g_{mj} \right), \tag{C.5}
\end{aligned}$$

where

$$S_n{}^m{}_{ij} = \left(\frac{\partial C^m_{ni}}{\partial y^j} - \frac{\partial C^m_{nj}}{\partial y^i} + C^h_{ni}C^m_{hj} - C^h_{nj}C^m_{hi} \right) F^2$$

and

$$S_{nmij} = g_{mt} S_n{}^t{}_{ij}.$$

Henceforth we assume the zero-degree homogeneity of the function $z(x, y)$ with respect to the argument y , having the identities

$$(\ln z)_{ni}l^i = -\frac{1}{F}(\ln z)_n, \quad (\ln z)_i l^i = 0. \tag{C.6}$$

By performing the contraction in (C.5), we get

$$\begin{aligned}
\frac{1}{z}(c_1(x))^2 F^{2a(x)} \tilde{R}_{nmij} l^m l^j &= \frac{1}{2F} \left(-(\ln z)_n l_i - (\ln z)_{ni} - (\ln z)_i l_n \right) \\
&+ \frac{1}{2z} \left(-\frac{a}{F}l_n + \frac{1}{2z}z_n \right) z_i + \frac{1}{2z} \frac{a}{F} z_i l_n - \frac{1}{4z^2} (z_h g^{hs} z_s) h_{in} + \frac{1}{2z} z_h C^h_{in},
\end{aligned}$$

or

$$\begin{aligned}
2\frac{1}{z}(c_1(x))^2 F^{2a(x)} \tilde{R}_{nmij} l^m l^j &= -\frac{1}{F}(\ln z)_n l_i - (\ln z)_{ni} - \frac{1}{F}(\ln z)_i l_n + \frac{1}{2z^2} z_n z_i \\
&\quad - \frac{1}{2z^2} (z_h g^{hs} z_s) h_{in} + \frac{1}{z} z_h C^h_{in}.
\end{aligned} \tag{C.7}$$

The vanishing

$$\tilde{R}_{nmij} = 0 \tag{C.8}$$

holds when

$$(\ln z)_{ni} = -\frac{1}{F}(\ln z)_n l_i - \frac{1}{F}(\ln z)_i l_n + \frac{1}{2z^2} z_n z_i - \frac{1}{2z^2} (z_h g^{hs} z_s) h_{in} + \frac{1}{z} z_h C^h_{in},$$

in which case from (C.5) we get

$$\begin{aligned}
\frac{1}{F^2} S_{nmij} &= -\frac{1}{F^2} (a^2 - 2a) (h_{nj} h_{mi} - h_{ni} h_{mj}) \\
&\quad + \frac{1}{2} \left(\frac{1}{F} (\ln z)_n l_j + \frac{1}{F} (\ln z)_j l_n - \frac{1}{2z^2} z_n z_j + \frac{1}{2z^2} (z_h g^{hs} z_s) h_{jn} \right) g_{mi} \\
&\quad - \frac{1}{2} \left(\frac{1}{F} (\ln z)_n l_i + \frac{1}{F} (\ln z)_i l_n - \frac{1}{2z^2} z_n z_i + \frac{1}{2z^2} (z_h g^{hs} z_s) h_{in} \right) g_{mj} \\
&\quad - \frac{1}{2} \left(\frac{1}{F} (\ln z)_m l_j + \frac{1}{F} (\ln z)_j l_m - \frac{1}{2z^2} z_m z_j + \frac{1}{2z^2} (z_h g^{hs} z_s) h_{jm} \right) g_{ni} \\
&\quad + \frac{1}{2} \left(\frac{1}{F} (\ln z)_m l_i + \frac{1}{F} (\ln z)_i l_m - \frac{1}{2z^2} z_m z_i + \frac{1}{2z^2} (z_h g^{hs} z_s) h_{im} \right) g_{nj} \\
&\quad + \frac{a}{F} \frac{1}{2z} \left(z_n (l_i h_{mj} - l_j h_{mi}) - z_m (l_i h_{nj} - l_j h_{ni}) \right) \\
&\quad - \frac{1}{2z} \left(-\frac{a}{F} l_n + \frac{1}{2z} z_n \right) (z_i g_{mj} - z_j g_{mi}) + \frac{1}{2z} \left(-\frac{a}{F} l_m + \frac{1}{2z} z_m \right) (z_i g_{jn} - z_j g_{in}) \\
&\quad + \frac{1}{4z^2} (z_h g^{hs} z_s) (g_{in} g_{mj} - g_{jn} g_{mi}).
\end{aligned}$$

Due simplifying yields

$$\begin{aligned}
& \frac{1}{F^2} S_{nmij} = -\frac{1}{F^2} (a^2 - 2a) (h_{nj} h_{mi} - h_{ni} h_{mj}) \\
& + \frac{1}{2F} \left((\ln z)_n l_j + (\ln z)_j l_n \right) g_{mi} - \frac{1}{2F} \left((\ln z)_n l_i + (\ln z)_i l_n \right) g_{mj} \\
& - \frac{1}{2F} \left((\ln z)_m l_j + (\ln z)_j l_m \right) g_{ni} + \frac{1}{2F} \left((\ln z)_m l_i + (\ln z)_i l_m \right) g_{nj} \\
& + \frac{1}{4z^2} (z_h g^{hs} z_s) \left[h_{jn} g_{mi} - h_{in} g_{mj} - h_{jm} g_{ni} + h_{im} g_{nj} \right] \\
& + \frac{a}{F} \frac{1}{2z} \left(z_n (l_i h_{mj} - l_j h_{mi}) - z_m (l_i h_{nj} - l_j h_{ni}) \right) \\
& + \frac{a}{F} \frac{1}{2z} \left[l_n (z_i h_{mj} - z_j h_{mi}) - l_m (z_i h_{jn} - z_j h_{in}) \right] + \frac{1}{4z^2} (z_h g^{hs} z_s) (g_{in} g_{mj} - g_{jn} g_{mi}) \\
& = -\frac{1}{F^2} (a^2 - 2a) (h_{nj} h_{mi} - h_{ni} h_{mj}) \\
& + \frac{a-1}{2zF} \left(z_n (l_i h_{mj} - l_j h_{mi}) - z_m (l_i h_{nj} - l_j h_{ni}) + l_n (z_i h_{mj} - z_j h_{mi}) - l_m (z_i h_{jn} - z_j h_{in}) \right) \\
& + \frac{1}{4z^2} (z_h g^{hs} z_s) \left(h_{jn} g_{mi} - h_{in} g_{mj} - h_{jm} g_{ni} + h_{im} g_{nj} + g_{in} g_{mj} - g_{jn} g_{mi} \right),
\end{aligned}$$

which is

$$\begin{aligned}
S_{nmij} &= a(2-a)(h_{nj} h_{mi} - h_{ni} h_{mj}) + F^2 \frac{1}{2z^2} (z_h g^{hs} z_s) (h_{nj} h_{mi} - h_{ni} h_{mj}) \\
& + \frac{a-1}{2z} \left(z_n (l_i h_{mj} - l_j h_{mi}) - z_m (l_i h_{nj} - l_j h_{ni}) + l_n (z_i h_{mj} - z_j h_{mi}) - l_m (z_i h_{jn} - z_j h_{in}) \right) F. \quad (\text{C.9})
\end{aligned}$$

Therefore, the known vanishing $S_{nmij} l^i = 0$ requires

$$(a-1) \left((\ln z)_n h_{mj} - (\ln z)_m h_{nj} \right) = 0. \quad (\text{C.10})$$

Whenever $a \neq 1$, we should take $z_n = 0$, which means that the *function z is independent of y* .

Lastly, it is worth noting that the case $a = 1$ would mean

$$S_{nmij} = h_{nj}h_{mi} - h_{ni}h_{mj}. \quad (\text{C.11})$$

In this case

$$\mathcal{C}_{\text{Ind.}} = 0 \quad (\text{C.12})$$

Appendix D: Evaluation of the coefficients N^k_{mni} in the \mathcal{F}^N -space

To evaluate the coefficients $N^k_{mni} = \partial N^k_{mn} / \partial y^i$, we use (II.3.32) and obtain

$$\begin{aligned} N^k_{mni} &= \frac{1}{F^2} l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2} (l^k h_{ni} + l_n h_i^k) \frac{\partial F}{\partial x^m} - \frac{1}{F} h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F} h_i^k \frac{\partial l_n}{\partial x^m} - l^k \frac{\partial \left(\frac{1}{F} h_{ni} \right)}{\partial x^m} \\ &\quad - \frac{\partial C^k_{ns}}{\partial y^i} N^s_m - C^k_{ns} N^s_{mi} - \frac{1}{F^2} l_i (l_n h_s^k - (1-H) l^k h_{ns}) N^s_m \\ &\quad + \frac{1}{F} (l_n h_s^k - (1-H) l^k h_{ns}) N^s_{mi} + \frac{1}{F^2} (h_{ni} h_s^k - l_n l^k h_{si} - l_n l_s h_i^k) N^s_m \\ &\quad - \frac{1}{F} (1-H) \left(\frac{1}{F} h_i^k h_{ns} + 2 l^k C_{nsi} \right) N^s_m + \frac{1}{F^2} (1-H) l^k l_n h_{si} N^s_m + \frac{1}{F^2} (1-H) l^k l_s h_{ni} N^s_m \\ &\quad - \left(y_{hp}^k t_i^p + \frac{H}{F} y_h^k l_i \right) F^H \left(\frac{\partial U_n^h}{\partial x^m} + L^h_{ms} U_n^s \right) - y_h^k F^H \left(\frac{\partial U_{ni}^h}{\partial x^m} + L^h_{ms} U_{ni}^s \right), \end{aligned}$$

or

$$\begin{aligned} N^k_{mni} &= \frac{1}{F^2} l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2} l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F} h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F} h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2} l^k h_{ni} \frac{\partial F}{\partial x^m} - \frac{1}{F} l^k \frac{\partial h_{ni}}{\partial x^m} \\ &\quad - \frac{\partial C^k_{ns}}{\partial y^i} N^s_m - C^k_{ns} N^s_{mi} - \frac{1}{F^2} l_i (l_n h_s^k - (1-H) l^k h_{ns}) N^s_m \\ &\quad + \frac{1}{F} (l_n h_s^k - (1-H) l^k h_{ns}) N^s_{mi} + \frac{1}{F^2} (h_{ni} h_s^k - l_n l^k h_{si} - l_n l_s h_i^k) N^s_m \\ &\quad - \frac{1}{F^2} (1-H) h_i^k h_{ns} N^s_m - \frac{2}{F} (1-H) l^k C_{nsi} N^s_m \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{F^2}(1-H)l^k l_n h_{si} N^s{}_m - H \frac{1}{F^2} l^k l_s h_{ni} N^s{}_m \\
& - \left(y_{hp}^k t_i^p + \frac{H}{F} y_h^k l_i \right) F^H \left(-N^s{}_m U_{ns}^h - N^s{}_{mn} U_s^h \right) - y_h^k F^H \left(\frac{\partial U_{ni}^h}{\partial x^m} + L^h{}_{ms} U_{ni}^s \right).
\end{aligned}$$

Reducing similar terms leads to

$$\begin{aligned}
N^k{}_{mni} &= \frac{1}{F^2} l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2} l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F} h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F} h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2} l^k h_{ni} \frac{\partial F}{\partial x^m} - \frac{1}{F} l^k \frac{\partial h_{ni}}{\partial x^m} \\
& - \frac{\partial C_{ns}^k}{\partial y^i} N^s{}_m - C_{ns}^k N^s{}_{mi} - \frac{1}{F^2} l_i (l_n h_s^k - (1-H)l^k h_{ns}) N^s{}_m \\
& + \frac{1}{F} (l_n h_s^k - (1-H)l^k h_{ns}) N^s{}_{mi} + \frac{1}{F^2} (h_{ni} h_s^k - l_n l^k h_{si} - l_n l_s h_i^k) N^s{}_m \\
& - \frac{1}{F^2} (1-H) h_i^k h_{ns} N^s{}_m - \frac{2}{F} (1-H) l^k C_{nsi} N^s{}_m \\
& + \frac{1}{F^2} (1-H) l^k l_n h_{si} N^s{}_m - H \frac{1}{F^2} l^k l_s h_{ni} N^s{}_m \\
& - y_p^k t_{iv}^p y_h^v F^H N^s{}_m U_{ns}^h - h_s^v y_p^k t_{iv}^p N^s{}_{mn} \\
& + \frac{H}{F} y_h^k l_i F^H (N^s{}_m U_{ns}^h + N^s{}_{mn} U_s^h) - y_h^k F^H \left(\frac{\partial U_{ni}^h}{\partial x^m} + L^h{}_{ms} U_{ni}^s \right).
\end{aligned}$$

Applying here (II.3.30) and (II.3.31) yields

$$\begin{aligned}
N^k{}_{mni} &= \frac{1}{F^2} l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2} l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F} h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F} h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2} l^k h_{ni} \frac{\partial F}{\partial x^m} - \frac{1}{F} l^k \frac{\partial h_{ni}}{\partial x^m} \\
& - \frac{\partial C_{ns}^k}{\partial y^i} N^s{}_m - C_{ns}^k N^s{}_{mi} - C_{is}^k N^s{}_{mn} - \frac{1}{F^2} l_i (l_n h_s^k - (1-H)l^k h_{ns}) N^s{}_m \\
& + \frac{1}{F} (l_n h_s^k - (1-H)l^k h_{ns}) N^s{}_{mi} + \frac{1}{F^2} (h_{ni} h_s^k - l_n l^k h_{si} - l_n l_s h_i^k) N^s{}_m \\
& - \frac{1}{F^2} (1-H) h_i^k h_{ns} N^s{}_m - \frac{2}{F} (1-H) l^k C_{nsi} N^s{}_m
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{F^2}(1-H)l^k l_n h_{si} N^s{}_m - H \frac{1}{F^2} l^k l_s h_{ni} N^s{}_m \\
& - \left(C^k{}_{iv} - (1-H) \frac{1}{F} (l_v \delta_i^k + l_i \delta_v^k - l^k g_{iv}) \right) y_h^v F^H U_{ns}^h N^s{}_m + (1-H) \frac{1}{F} (l_i h_s^k - l^k h_{is}) N^s{}_{mn} \\
& + \frac{H}{F} y_h^k l_i F^H (N^s{}_m U_{ns}^h + N^s{}_{mn} U_s^h) - y_h^k F^H \left(\frac{\partial U_{ni}^h}{\partial x^m} + L^h{}_{ms} U_{ni}^s \right).
\end{aligned}$$

From the representation

$$U_n^h = \frac{1}{F^H} t_n^h - \frac{1}{F} H U^h l_n$$

(see (II.3.5)) let us evaluate the coefficients

$$U_{ni}^h = \frac{\partial U_n^h}{\partial y^i}.$$

We get

$$\begin{aligned}
U_{ni}^h &= -\frac{H}{F} l_i \frac{1}{F^H} t_n^h + \frac{1}{F^H} t_{ni}^h + \frac{1}{F^2} l_i H U^h l_n - \frac{1}{F} H U_i^h l_n - \frac{1}{F^2} H U^h h_{ni} \\
&= -\frac{H}{F} (l_i U_n^h + l_n U_i^h) + \frac{1}{F^H} t_{ni}^h + \frac{1}{F^2} (1-H) H U^h l_n l_i - \frac{1}{F^2} H U^h h_{ni}.
\end{aligned}$$

Here,

$$t_{ni}^h = C^s{}_{ni} t_s^h - (1-H) \frac{1}{F} \left(l_i t_n^h + l_n t_i^h - \frac{H}{F} t^h g_{ni} \right)$$

(see (II.3.30)). From this it follows that

$$\begin{aligned}
U_{ni}^h &= -\frac{H}{F} (l_i U_n^h + l_n U_i^h) + \frac{1}{F^H} \left[C^s{}_{ni} t_s^h - (1-H) \frac{1}{F} (l_i t_n^h + l_n t_i^h) \right] \\
&+ \frac{1}{F^2} \frac{1}{F^H} (1-H) H t^h g_{ni} - \frac{1}{F^2} H U^h (g_{ni} - (2-H) l_n l_i).
\end{aligned}$$

We have here

$$\begin{aligned}
U_{ni}^h &= -\frac{H}{F} (l_i U_n^h + l_n U_i^h) + \frac{1}{F^H} \left[C^s{}_{ni} t_s^h - \frac{1}{F} (l_i t_n^h + l_n t_i^h) \right] + \frac{1}{F^H} H \frac{1}{F} (l_i t_n^h + l_n t_i^h) \\
&- \frac{1}{F^2} \frac{1}{F^H} H^2 t^h g_{ni} + \frac{2-H}{F^2} H U^h l_n l_i \\
&= C^s{}_{ni} t_s^h \frac{1}{F^H} - \frac{1}{F} \frac{1}{F^H} (l_i t_n^h + l_n t_i^h) - \frac{1}{F^2} H^2 U^h g_{ni} + \frac{2+H}{F^2} H U^h l_n l_i.
\end{aligned}$$

Thus it is valid that

$$U_{ni}^h = C_{ni}^s t_s^h \frac{1}{FH} - \frac{1}{F} (l_i U_n^h + l_n U_i^h) - \frac{1}{F^2} H^2 U^h h_{ni}. \quad (D.1)$$

We may readily deduce the contraction

$$y_h^v F^H U_{ns}^h = C_{sn}^v - \frac{1}{F} (l_s h_n^v + l_n h_s^v) - \frac{1}{F} H h_{ns} l^v. \quad (D.2)$$

Inserting yields

$$\begin{aligned} N^k{}_{mni} &= \frac{1}{F^2} l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2} l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F} h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F} h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2} l^k h_{ni} \frac{\partial F}{\partial x^m} - \frac{1}{F} l^k \frac{\partial h_{ni}}{\partial x^m} \\ &\quad - \frac{\partial C_{ns}^k}{\partial y^i} N^s{}_m - C_{ns}^k N^s{}_{mi} - C_{is}^k N^s{}_{mn} - \frac{1}{F^2} l_i (l_n h_s^k - (1-H) l^k h_{ns}) N^s{}_m \\ &\quad + \frac{1}{F} (l_n h_s^k - (1-H) l^k h_{ns}) N^s{}_{mi} + \frac{1}{F^2} (h_{ni} h_s^k - l_n l^k h_{si} - l_n l_s h_i^k) N^s{}_m \\ &\quad - \frac{1}{F^2} (1-H) h_i^k h_{ns} N^s{}_m - \frac{2}{F} (1-H) l^k C_{nsi} N^s{}_m \\ &\quad + \frac{1}{F^2} (1-H) l^k l_n h_{si} N^s{}_m - H \frac{1}{F^2} l^k l_s h_{ni} N^s{}_m \\ &\quad - \left(C_{iv}^k - (1-H) \frac{1}{F} (l_v \delta_i^k + l_i \delta_v^k - l^k g_{iv}) \right) \left[C_{sn}^v - \frac{1}{F} (l_s h_n^v + l_n h_s^v) - \frac{1}{F} H h_{ns} l^v \right] N^s{}_m \\ &\quad + (1-H) \frac{1}{F} (l_i h_s^k - l^k h_{is}) N^s{}_{mn} + \frac{H}{F} l_i \left[C_{sn}^k - \frac{1}{F} (l_s h_n^k + l_n h_s^k) - \frac{H}{F} h_{ns} l^k \right] N^s{}_m + \frac{H}{F} l_i N^s{}_{mn} h_s^k \\ &\quad + y_h^k F^H \frac{\partial \frac{1}{F} (l_i U_n^h + l_n U_i^h)}{\partial x^m} + y_h^k F^H \frac{\partial \frac{1}{F^2} H^2 U^h h_{ni}}{\partial x^m} \\ &\quad - \frac{\partial C_{ni}^k}{\partial x^m} - C_{ni}^s y_h^k F^H \frac{\partial \left(U_s^h + \frac{1}{F} H U^h l_s \right)}{\partial x^m} - y_h^k F^H L^h{}_{ms} U_{ni}^s, \end{aligned}$$

or

$$\begin{aligned}
N^k_{mni} &= \frac{1}{F^2} l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2} l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F} h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F} h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2} l^k h_{ni} \frac{\partial F}{\partial x^m} - \frac{1}{F} l^k \frac{\partial h_{ni}}{\partial x^m} \\
&- \frac{\partial C^k_{ni}}{\partial x^m} - \frac{\partial C^k_{ns}}{\partial y^i} N^s_m - C^k_{ns} N^s_{mi} - C^k_{is} N^s_{mn} - \frac{1}{F^2} l_i (l_n h_s^k - (1-H) l^k h_{ns}) N^s_m \\
&+ \frac{1}{F} (l_n h_s^k - (1-H) l^k h_{ns}) N^s_{mi} + \frac{1}{F^2} (h_{ni} h_s^k - l_n l^k h_{si} - l_n l_s h_i^k) N^s_m \\
&- \frac{1}{F^2} (1-H) h_i^k h_{ns} N^s_m - \frac{2}{F} (1-H) l^k C_{nsi} N^s_m \\
&+ \frac{1}{F^2} (1-H) l^k l_n h_{si} N^s_m - H \frac{1}{F^2} l^k l_s h_{ni} N^s_m \\
&- \left[C^v_{sn} C^k_{iv} - \frac{1}{F} (l_s C^k_{in} + l_n C^k_{is}) \right] N^s_m - (1-H) \frac{1}{F^2} \delta_i^k H h_{ns} N^s_m \\
&+ (1-H) \frac{1}{F} (l_i \delta_v^k - l^k g_{iv}) \left[C^v_{sn} - \frac{1}{F} (l_s h_n^v + l_n h_s^v) \right] N^s_m - (1-H) \frac{1}{F} l^k h_{is} N^s_{mn} \\
&+ \frac{H}{F} l_i \left[C^k_{sn} - \frac{1}{F} (l_s h_n^k + l_n h_s^k) - \frac{H}{F} h_{ns} l^k \right] N^s_m + \frac{1}{F} l_i N^s_{mn} h_s^k \\
&+ y_h^k F^H \frac{\partial \frac{1}{F} (l_i U_n^h + l_n U_i^h)}{\partial x^m} + y_h^k F^H \frac{1}{F^2} H^2 h_{ni} \frac{\partial U^h}{\partial x^m} + y^k H \frac{\partial \frac{1}{F^2} h_{ni}}{\partial x^m} + 2y^k H_m \frac{1}{F^2} h_{ni} \\
&- C^s_{ni} y_h^k F^H \frac{\partial \left(\frac{1}{F} H U^h l_s \right)}{\partial x^m} - y_h^k F^H L^h_{sm} U^s_{ni} - C^v_{ni} y_h^k F^H \frac{\partial U^h_v}{\partial x^m}.
\end{aligned}$$

Finally we apply here (II.3.32), obtaining

$$\begin{aligned}
N^k_{mni} &= \frac{1}{F^2} l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2} l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F} h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F} h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2} l^k h_{ni} \frac{\partial F}{\partial x^m} - \frac{1}{F} l^k \frac{\partial h_{ni}}{\partial x^m} \\
&- \frac{\partial C^k_{ni}}{\partial x^m} - \frac{\partial C^k_{ns}}{\partial y^i} N^s_m - C^k_{ns} N^s_{mi} - C^k_{is} N^s_{mn} + N^k_{ms} C^s_{ni}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{F^2}l_i(l_n h_s^k - (1-H)l^k h_{ns})N^s_m + \frac{1}{F^2}l_n(1-H)l^k h_{is}N^s_m \\
& + \frac{1}{F}h_s^k(l_n N^s_{mi} + l_i N^s_{mn}) - \frac{1}{F}(1-H)l^k(h_{ns}N^s_{mi} + h_{is}N^s_{mn}) \\
& + \frac{1}{F^2}(h_{ni}h_s^k - l_n l^k h_{si} - l_n l_s h_i^k)N^s_m \\
& - \frac{1}{F^2}(1-H)h_i^k h_{ns}N^s_m - \frac{2}{F}(1-H)l^k C_{nsi}N^s_m - H\frac{1}{F^2}l^k l_s h_{ni}N^s_m \\
& - \left[C^v_{sn}C^k_{iv} - \frac{1}{F}(l_s C^k_{in} + l_n C^k_{is}) \right] N^s_m - (1-H)\frac{1}{F^2}\delta_i^k H h_{ns}N^s_m \\
& + (1-H)\frac{1}{F}l_i \left[C^k_{sn} - \frac{1}{F}(l_s h_n^k + l_n h_s^k) \right] N^s_m - (1-H)\frac{1}{F}l^k \left[C_{isn} - \frac{1}{F}(l_s h_{in} + l_n h_{is}) \right] N^s_m \\
& + \frac{H}{F}l_i \left[C^k_{sn} - \frac{1}{F}(l_s h_n^k + l_n h_s^k) - \frac{H}{F}h_{ns}l^k \right] N^s_m \\
& + y_h^k F^H \frac{\partial \frac{1}{F}(l_i U_n^h + l_n U_i^h)}{\partial x^m} + y_h^k F^H \frac{1}{F^2} H^2 h_{ni} \frac{\partial U^h}{\partial x^m} + y^k H \frac{\partial \frac{1}{F^2} h_{ni}}{\partial x^m} + 2y^k H_m \frac{1}{F^2} h_{ni} \\
& - C^s_{ni} l^k \frac{\partial l_s}{\partial x^m} - y_h^k F^H L^h_{sm} U^s_{ni} \\
& - C^v_{ni} \left[-\frac{1}{F}h_v^k \frac{\partial F}{\partial x^m} - l^k \frac{\partial l_v}{\partial x^m} - C^k_{vs} N^s_m - \frac{1}{F}(1-H)l^k h_{vs} N^s_m \right] + C^v_{ni} y_h^k F^H L^h_{sm} U^s_v.
\end{aligned}$$

Reducing similar terms yields now

$$\begin{aligned}
N^k_{mni} &= \frac{1}{F^2}l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2}l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F}h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F}h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2}l^k h_{ni} \frac{\partial F}{\partial x^m} \\
& - (1-H)\frac{1}{F}l^k \left[\frac{\partial h_{ni}}{\partial x^m} + 2C_{nsi}N^s_m + h_{ns}N^s_{mi} + h_{is}N^s_{mn} \right] - \mathcal{D}_m C^k_{ni}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{F^2}l_i(l_n h_s^k - (1-H)l^k h_{ns})N^s_m + \frac{1}{F^2}l_n(1-H)l^k h_{is}N^s_m + \frac{1}{F}h_s^k(l_n N^s_{mi} + l_i N^s_{mn}) \\
& + \frac{1}{F^2}(h_{ni}h_s^k - l_n l^k h_{si} - l_n l_s h_i^k)N^s_m - \frac{1}{F^2}(1-H)h_i^k h_{ns}N^s_m - H\frac{1}{F^2}l^k l_s h_{ni}N^s_m \\
& - \left[C^v_{sn} C^k_{iv} - \frac{1}{F}(l_i C^k_{ns} + l_n C^k_{is}) \right] N^s_m - (1-H)\frac{1}{F^2}h_i^k H h_{ns}N^s_m - (1-H)\frac{1}{F^2}l^k l_i H h_{ns}N^s_m \\
& - \frac{1}{F^2}l_i(l_s h_n^k + l_n h_s^k)N^s_m - (1-H)\frac{1}{F}l^k \left[C_{isn} - \frac{1}{F}(l_s h_{in} + l_n h_{is}) \right] N^s_m - \frac{H}{F^2}l_i l^k H h_{ns}N^s_m
\end{aligned}$$

$$+ y_h^k F^H \frac{\partial \frac{1}{F}(l_i U_n^h + l_n U_i^h)}{\partial x^m} + y_h^k F^H \frac{1}{F^2} H^2 h_{ni} \frac{\partial U^h}{\partial x^m} + y^k H h_{ni} \frac{\partial \frac{1}{F^2}}{\partial x^m} + 2y^k H_m \frac{1}{F^2} h_{ni}$$

$$- y_h^k F^H L^h_{sm} U_{ni}^s - C^v_{ni} \left[-C^k_{vs} N^s_m - \frac{1}{F}(1-H)l^k h_{vs} N^s_m \right] + C^v_{ni} y_h^k F^H L^h_{sm} U_v^s.$$

The next step is to transform the representation to

$$\begin{aligned}
N^k_{mni} &= \frac{1}{F^2}l_i h_n^k \frac{\partial F}{\partial x^m} + \frac{1}{F^2}l_n h_i^k \frac{\partial F}{\partial x^m} - \frac{1}{F}h_n^k \frac{\partial l_i}{\partial x^m} - \frac{1}{F}h_i^k \frac{\partial l_n}{\partial x^m} + \frac{1}{F^2}l^k h_{ni} \frac{\partial F}{\partial x^m} \\
& - (1-H)\frac{1}{F}l^k \left[\frac{\partial h_{ni}}{\partial x^m} + 2C_{nsi}N^s_m + h_{ns}N^s_{mi} + h_{is}N^s_{mn} \right] - \mathcal{D}_m C^k_{ni} \\
& - \frac{1}{F^2}l_i(l_n h_s^k - (1-H)l^k h_{ns})N^s_m + \frac{1}{F^2}l_n(1-H)l^k h_{is}N^s_m + \frac{1}{F}h_s^k(l_n N^s_{mi} + l_i N^s_{mn}) \\
& + \frac{1}{F^2}(h_{ni}h_s^k - l_n l^k h_{si} - l_n l_s h_i^k - l_i l_s h_n^k)N^s_m - \frac{1}{F^2}(1-H)h_i^k h_{ns}N^s_m - H\frac{1}{F^2}l^k l_s h_{ni}N^s_m \\
& - \left[\frac{1}{F^2}S_n^k{}_{si} - \frac{1}{F}(l_i C^k_{ns} + l_n C^k_{is}) \right] N^s_m - (1-H)\frac{1}{F^2}h_i^k H h_{ns}N^s_m
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{F^2}l^k l_i H h_{ns} N^s_m - \frac{1}{F^2}l_i l_n h_s^k N^s_m + (1-H)\frac{1}{F^2}l^k l_s h_{in} N^s_m + (1-H)\frac{1}{F^2}l^k l_n h_{is} N^s_m \\
& + y_h^k F^H \frac{1}{F} \left(l_i \frac{\partial U_n^h}{\partial x^m} + l_n \frac{\partial U_i^h}{\partial x^m} \right) + h_n^k \frac{\partial \frac{1}{F} l_i}{\partial x^m} + h_i^k \frac{\partial \frac{1}{F} l_n}{\partial x^m} \\
& + y_h^k F^H \frac{1}{F^2} H^2 h_{ni} \frac{\partial U^h}{\partial x^m} - \frac{2}{F^2} l^k H h_{ni} \frac{\partial F}{\partial x^m} + 2 \frac{1}{F} l^k H_m h_{ni} - y_h^k F^H L^h_{sm} U_{ni}^s + C^v_{ni} y_h^k F^H L^h_{sm} U_v^s,
\end{aligned}$$

where

$$S_n^k{}_{si} = F^2 (C^v_{in} C^k_{sv} - C^v_{sn} C^k_{iv}).$$

We come to

$$\begin{aligned}
N^k_{mni} &= -(1-H) \frac{1}{F} l^k \left[\frac{\partial h_{ni}}{\partial x^m} + 2 C_{nsi} N^s_m + h_{ns} N^s_{mi} + h_{is} N^s_{mn} \right] - \mathcal{D}_m C^k_{ni} \\
& - \frac{1}{F^2} l_i (l_n h_s^k - (1-H) l^k h_{ns}) N^s_m + \frac{1}{F^2} l_n (1-H) l^k h_{is} N^s_m \\
& + \frac{1}{F} h_s^k (l_n N^s_{mi} + l_i N^s_{mn}) + \frac{1}{F^2} (h_{ni} h_s^k - l_n l_s h_i^k - l_i l_s h_n^k) N^s_m \\
& - \left[\frac{1}{F^2} S_n^k{}_{si} - \frac{1}{F} (l_i C^k_{ns} + l_n C^k_{is}) \right] N^s_m - (1-H^2) \frac{1}{F^2} h_i^k h_{ns} N^s_m \\
& - \frac{1}{F^2} l_i l_n h_s^k N^s_m - H \frac{1}{F^2} l^k (l_i h_{ns} + l_n h_{is}) N^s_m + y_h^k F^H \frac{1}{F} \left(l_i \frac{\partial U_n^h}{\partial x^m} + l_n \frac{\partial U_i^h}{\partial x^m} \right) \\
& + y_h^k F^H \frac{1}{F^2} H^2 h_{ni} \frac{\partial U^h}{\partial x^m} + 2 \frac{1}{F} l^k H_m h_{ni} + y_h^k F^H L^h_{sm} \left[\frac{1}{F} (l_i U_n^s + l_n U_i^s) + \frac{H^2}{F^2} U^s h_{ni} \right] \\
& = -(1-H) \frac{1}{F} l^k \mathcal{D}_m h_{ni} - \mathcal{D}_m C^k_{ni} - \frac{1}{F^2} l_i l_n h_s^k N^s_m + \frac{1}{F^2} (h_{ni} h_s^k - l_n l_s h_i^k - l_i l_s h_n^k) N^s_m \\
& - \left[\frac{1}{F^2} S_n^k{}_{si} - \frac{1}{F} (l_i C^k_{ns} + l_n C^k_{is}) \right] N^s_m - (1-H^2) \frac{1}{F^2} h_i^k h_{ns} N^s_m
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{F^2} l_i l_n h_s^k N^s{}_m - H \frac{1}{F^2} l^k (l_i h_{ns} + l_n h_{is}) N^s{}_m \\
& + y_h^k F^H \frac{1}{F} \left[l_i \left(\frac{\partial U_n^h}{\partial x^m} + U_s^h N^s{}_{mn} + L^h{}_{sm} U_n^s \right) + l_n \left(\frac{\partial U_i^h}{\partial x^m} + U_s^h N^s{}_{mi} + L^h{}_{sm} U_i^s \right) \right] \\
& + y_h^k F^H \frac{1}{F^2} H^2 h_{ni} \left(\frac{\partial U^h}{\partial x^m} + L^h{}_{sm} U_m^s \right) + 2 \frac{1}{F} l^k H_m h_{ni}.
\end{aligned}$$

Use here the equality

$$\mathcal{D}_m h_{ni} = -\frac{2}{H} H_m h_{ni}$$

(see (II.3.21)).

The rest is

$$\begin{aligned}
N^k{}_{mni} &= \frac{2}{H} H_m \frac{1}{F} l^k h_{ni} - \mathcal{D}_m C^k{}_{ni} - \frac{1}{F^2} l_i l_n h_s^k N^s{}_m + \frac{1}{F^2} (h_{ni} h_s^k - l_n l_s h_i^k - l_i l_s h_n^k) N^s{}_m \\
& - \left[\frac{1}{F^2} S_n{}^k{}_{si} - \frac{1}{F} (l_i C^k{}_{ns} + l_n C^k{}_{is}) \right] N^s{}_m - (1 - H^2) \frac{1}{F^2} h_i^k h_{ns} N^s{}_m - \frac{1}{F^2} l_i l_n h_s^k N^s{}_m \\
& - H \frac{1}{F^2} l^k (l_i h_{ns} + l_n h_{is}) N^s{}_m - y_h^k F^H \frac{1}{F} (l_i U_{ns}^h + l_n U_{is}^h) N^s{}_m - y_h^k F^H \frac{1}{F^2} H^2 h_{ni} U_s^h N^s{}_m \\
& = \frac{2}{H} H_m \frac{1}{F} l^k h_{ni} - \mathcal{D}_m C^k{}_{ni} - \frac{1}{F^2} l_i l_n h_s^k N^s{}_m - \frac{1}{F^2} (l_n l_s h_i^k + l_i l_s h_n^k) N^s{}_m \\
& - \left[\frac{1}{F^2} S_n{}^k{}_{si} - \frac{1}{F} (l_i C^k{}_{ns} + l_n C^k{}_{is}) \right] N^s{}_m - (1 - H^2) \frac{1}{F^2} (h_i^k h_{ns} - h_{in} h_s^k) N^s{}_m \\
& - \frac{1}{F^2} l_i l_n h_s^k N^s{}_m - H \frac{1}{F^2} l^k (l_i h_{ns} + l_n h_{is}) N^s{}_m \\
& - \frac{1}{F} \left[l_i \left(C^k{}_{sn} - \frac{1}{F} (l_s h_n^k + l_n h_s^k) - \frac{H}{F} h_{ns} l^k \right) + l_n \left(C^k{}_{si} - \frac{1}{F} (l_s h_i^k + l_i h_s^k) - \frac{H}{F} h_{is} l^k \right) \right] N^s{}_m.
\end{aligned}$$

Since the indicatrix curvature equals H^2 , we have $S_n{}^k{}_{si} = -(1 - H^2) (h_i^k h_{ns} - h_{in} h_s^k)$.

The eventual result is the representation

$$N^k_{mni} = \frac{2}{H} H_m \frac{1}{F} l^k h_{ni} - \mathcal{D}_m C^k_{ni}. \quad (\text{D.3})$$

Thus, Proposition II.3.5 is valid.

Appendix E: Implications from angle

Below, the consideration refers to an *arbitrary* Finsler space. No assumptions concerning the curvature of indicatrix are made. We use the angle $\alpha = \alpha_{\{x\}}(y_1, y_2)$ which is the geodesic-arc distance on the indicatrix, in accordance with the initial definition (I.1.1).

In terms of the function

$$E := \frac{1}{2} \alpha^2 \quad (\text{E.1})$$

the preservation equation $d_i \alpha + (1/H) H_i \alpha = 0$ proposed by (I.1.28) reads

$$\frac{\partial E}{\partial x^i} + N^k_{1i} \frac{\partial E}{\partial y_1^k} + N^k_{2i} \frac{\partial E}{\partial y_2^k} = -\frac{2}{H} H_i E, \quad (\text{E.2})$$

where $N^k_{1i} = N^k_i(x, y_1)$, $N^k_{2i} = N^k_i(x, y_2)$, and $H_i = \partial H / \partial x^i$. Henceforth,

$$H = H(x). \quad (\text{E.3})$$

We are aimed to extract the required coincidence limits from the function E , treating the indicatrix naturally to be a particular Riemannian space metricized by the help of the metric tensor induced by the Finsler metric tensor.

Let a set of scalars $u^a = u^a(x, y)$ be used to coordinatize the indicatrices; the indices a, b, \dots will be specified over the range $(1, 2, \dots, N-1)$. We shall use the derivative objects

$$u_m^a = \frac{\partial u^a}{\partial y^m}, \quad u_{mk}^a = \frac{\partial u_m^a}{\partial y^k}.$$

The scalars are assumed to be positively homogeneous of degree zero with respect to the variable y :

$$u^a(x, ky) = u^a(x, y), \quad k > 0, \quad \forall y, \quad (\text{E.4})$$

which directly entails the identities

$$u_m^a y^m = 0, \quad u_{mk}^a y^k = -u_m^a. \quad (\text{E.5})$$

Using the parametrical representation $l^i = t^i(u^a)$ of the indicatrix, where l^i are unit vectors (possessing the property $F(l) = 1$), we can construct the induced metric tensor

$$i_{ab}(u^c) = g_{mn} t_a^m t_b^n \equiv h_{mn} t_a^m t_b^n \quad (\text{E.6})$$

on the indicatrix by the help of the projection factors $t_a^m = \partial t^m / \partial u^a$ (the method was described in detail in Section 5.8) of [1]).

The validity of the equalities

$$F u_m^b t_c^m = \delta_c^b, \quad F u_m^b t_b^k = h_m^k, \quad F u_k^c = g_{km} t_a^m i^{ac}, \quad t_e^n i^{ec} t_c^i = h^{ni}, \quad \frac{1}{F^2} h_{mn} = i_{ab} u_m^a u_n^b \quad (\text{E.7})$$

can readily be verified.

From the identity $l_m t_a^m = 0$ it follows that

$$l_m t_{ab}^m = -i_{ab}, \quad (\text{E.8})$$

where $t_{ab}^m = \partial t_a^m / \partial u^b$.

From (E.6) we get

$$i_{ab,c} = 2FC_{mnk} t_a^m t_b^n t_c^k + g_{nm} (t_{ac}^m t_b^n + t_a^m t_{bc}^n).$$

With the coefficients $i_{ac,b} = \partial i_{ac} / \partial u^b$ we obtain

$$(i_{ac,b} + i_{bc,a} - i_{ab,c}) = 2FC_{mnk} t_a^m t_b^n t_c^k + 2g_{nm} t_{ab}^m t_c^n,$$

which entails

$$(i_{ae,b} + i_{be,a} - i_{ab,e}) i^{ec} = 2FC_{mnk} t_a^m t_b^n t_e^k i^{ec} + 2g_{nm} t_{ab}^m t_e^n i^{ec},$$

so that

$$t_c^i \left(i_{ab}^c - FC_{mnk} t_a^m t_b^n t_e^k i^{ec} \right) = h_m^i t_{ab}^m$$

and

$$t_{ab}^i = t_c^i \left(i_{ab}^c - FC_{mnk} t_a^m t_b^n t_e^k i^{ec} \right) - l^i i_{ab} \quad (\text{E.9})$$

(this equation is equivalent to (5.8.8) of [1]).

The indicatrix Christoffel symbols

$$i_{ab}^c = \frac{1}{2} i^{ce} \left(\frac{\partial i_{ea}}{\partial u^b} + \frac{\partial i_{eb}}{\partial u^a} - \frac{\partial i_{ab}}{\partial u^e} \right)$$

and the indicatrix curvature tensor

$$I_a{}^e{}_{bd} := \frac{\partial i_{ab}^e}{\partial u^d} - \frac{\partial i_{ad}^e}{\partial u^b} + i_{ab}^f i_{fd}^e - i_{ad}^f i_{fb}^e \quad (\text{E.10})$$

will be used.

Constructing the tensor

$$S_{abcd} = -\frac{1}{3} (I_{acbd} + I_{adbc}), \quad (\text{E.11})$$

where $I_{acbd} = I_a{}^e{}_{bd} i_{ec}$, we obtain the useful identity

$$S_{abcd} - S_{acbd} = -I_{adbc}. \quad (\text{E.12})$$

It follows that

$$i_{ab} (u_{mj}^b + i_{cv}^b u_m^v u_j^c) = h_{pq} (t_a^p t_b^q u_{mj}^b + i_{cv}^b t_a^p t_b^q u_m^v u_j^c) = h_{pq} t_a^p \left(\frac{\partial \frac{1}{F} h_m^q}{\partial y^j} - u_m^b t_{bc}^q u_j^c + i_{cv}^b t_b^q u_m^v u_j^c \right).$$

Taking t_{ab}^p from (E.9), we get

$$i_{ab} (u_{mj}^b + i_{cv}^b u_m^v u_j^c) = h_{pq} t_a^p \left(\frac{\partial \frac{1}{F} h_m^q}{\partial y^j} - u_m^b t_f^q \left(i_{bc}^f - FC_{rsk} t_b^r t_c^s t_e^k i^{ef} \right) u_j^c + i_{cv}^b t_b^q u_m^v u_j^c \right).$$

Here,

$$t_e^k i^{ef} = F u_h^f g^{hk},$$

so that

$$i_{ab} (u_{mj}^b + i^b_{cv} u_m^v u_j^c) = h_{pq} t_a^p \left(\frac{\partial \frac{1}{F} h_m^q}{\partial y^j} + \frac{1}{F} C_{mj}^q \right), \quad (\text{E.13})$$

from which it follows that

$$u_{mj}^b + i^b_{cv} u_m^v u_j^c = F u_t^b \left(\frac{\partial \frac{1}{F} h_m^t}{\partial y^j} + \frac{1}{F} C_{mj}^t \right) \quad (\text{E.14})$$

and

$$u_{mk}^a u_n^b i_{ab} + u_m^a u_k^f u_n^b i_{af} i_{cb} = -\frac{1}{F^3} (h_{nk} l_m + h_{nm} l_k) + \frac{1}{F^2} C_{kmn}. \quad (\text{E.15})$$

Now we consider the quantity (E.1) on the indicatrix:

$$E = M(x, u_1, u_2), \quad (\text{E.16})$$

where M is a scalar.

There arise the objects

$$M_{1a2c} \stackrel{\text{def}}{=} \frac{\partial M_{1a}}{\partial u_2^c}, \quad M_{1a2c2d} \stackrel{\text{def}}{=} \frac{\partial M_{1a2c}}{\partial u_2^d} - i^{2f}_{2d2c} M_{1a2f}, \quad (\text{E.17})$$

together with

$$M_{1a1c} \stackrel{\text{def}}{=} \frac{\partial M_{1a}}{\partial u_1^c} - i^{1f}_{1a1c} M_{1f}, \quad M_{1a1b2c} \stackrel{\text{def}}{=} \frac{\partial M_{1a1b}}{\partial u_2^c} = \frac{\partial^3 M}{\partial u_1^a \partial u_1^b \partial u_2^c} - i^{1f}_{1a1b} M_{1f2c} \quad (\text{E.18})$$

$$M_{1a1b2c2d} \stackrel{\text{def}}{=} \frac{\partial M_{1a1b2c}}{\partial u_2^d} - i^{2f}_{2d2c} M_{1a1b2f} = \frac{\partial^4 M}{\partial u_1^a \partial u_1^b \partial u_2^c \partial u_2^d} - i^{1f}_{1a1b} \frac{\partial M_{1f2c}}{\partial u_2^d} - i^{2f}_{2d2c} M_{1a1b2f}. \quad (\text{E.19})$$

It follows that

$$\frac{\partial^4 M}{\partial u_1^a \partial u_1^b \partial u_2^c \partial u_2^d} = M_{1a1b2c2d} + i^{1f}_{1a1b} \frac{\partial M_{1f2c}}{\partial u_2^d} + i^{2f}_{2d2c} M_{1a1b2f}. \quad (\text{E.20})$$

In the limit $u_2 \rightarrow u_1$ we have

$$\frac{\partial M}{\partial u_1^a} \rightarrow 0, \quad \frac{\partial M}{\partial u_2^a} \rightarrow 0, \quad (\text{E.21})$$

and

$$\frac{\partial^2 M}{\partial u_1^a \partial u_1^b} \rightarrow i_{ab}, \quad \frac{\partial^2 M}{\partial u_1^a \partial u_2^b} \rightarrow -i_{ab}, \quad \frac{\partial^2 M}{\partial u_2^a \partial u_2^b} \rightarrow i_{ab}, \quad (\text{E.22})$$

together with

$$\frac{\partial^3 M}{\partial u_1^a \partial u_2^b \partial u_2^c} \rightarrow -i^e{}_{bc} i_{ea}, \quad \frac{\partial^3 M}{\partial u_1^a \partial u_1^b \partial u_2^c} \rightarrow -i^f{}_{ab} i_{fc} \quad (\text{E.23})$$

(see Section 3.2 in [12]).

Also,

$$M_{1a1b2c2d} \rightarrow S_{abcd} \quad (\text{E.24})$$

(see (3.2.69) in [12]). From (E.20) it follows that

$$\frac{\partial^4 M}{\partial u_1^a \partial u_1^b \partial u_2^c \partial u_2^d} \rightarrow S_{abcd} - i^f{}_{ab} i^e{}_{dc} i_{fe}. \quad (\text{E.25})$$

We find

$$\frac{\partial E}{\partial y_2^k} = u_{2k}^d \frac{\partial M}{\partial u_2^d}, \quad \frac{\partial^2 E}{\partial y_2^k \partial y_2^n} = u_{2k2n}^d \frac{\partial M}{\partial u_2^d} + u_{2k}^d \frac{\partial^2 M}{\partial u_2^d \partial u_2^t} u_{2n}^t \quad (\text{E.26})$$

and

$$\frac{\partial^3 E}{\partial y_2^k \partial y_2^n \partial y_1^m} = u_{2k2n}^d \frac{\partial^2 M}{\partial u_2^d \partial u_1^v} u_{1m}^v + u_{2k}^d \frac{\partial^3 M}{\partial u_2^d \partial u_2^t \partial u_1^s} u_{2n}^t u_{1m}^s. \quad (\text{E.27})$$

Moreover,

$$\begin{aligned} \frac{\partial^4 E}{\partial y_2^k \partial y_2^n \partial y_1^m \partial y_1^j} &= u_{2k2n}^d \frac{\partial^3 M}{\partial u_2^d \partial u_1^v \partial u_1^s} u_{1m}^v u_{1j}^s + u_{2k2n}^d \frac{\partial^2 M}{\partial u_2^d \partial u_1^v} u_{1m1j}^v \\ &+ u_{2k}^d \frac{\partial^3 M}{\partial u_2^d \partial u_2^t \partial u_1^s} u_{2n}^t u_{1m1j}^s + u_{2k}^d \frac{\partial^4 M}{\partial u_2^d \partial u_2^t \partial u_1^s \partial u_1^t} u_{2n}^t u_{1m}^s u_{1j}^t. \end{aligned} \quad (\text{E.28})$$

These observations entail the limits

$$y_2 \rightarrow y_1 : \quad \frac{\partial E}{\partial y_1^m} \rightarrow 0, \quad \frac{\partial E}{\partial y_2^m} \rightarrow 0, \quad (\text{E.29})$$

$$\frac{\partial^2 E}{\partial y_1^m \partial y_1^n} \rightarrow \frac{1}{F^2} h_{mn}, \quad \frac{\partial^2 E}{\partial y_2^m \partial y_2^n} \rightarrow \frac{1}{F^2} h_{mn}, \quad \frac{\partial^2 E}{\partial y_1^m \partial y_2^n} \rightarrow -\frac{1}{F^2} h_{mn}, \quad (\text{E.30})$$

and

$$\frac{\partial^3 E}{\partial y_2^k \partial y_1^m \partial y_2^n} \rightarrow -(u_{nk}^a u_m^b i_{ab} + u_m^a u_k^b u_n^c i^e{}_{bc} i_{ea}), \quad (\text{E.31})$$

plus

$$\frac{\partial^3 E}{\partial y_1^k \partial y_1^m \partial y_2^n} \rightarrow -(u_{mk}^a u_n^b i_{ab} + u_n^a u_k^b u_m^c i^e{}_{bc} i_{ea}), \quad (\text{E.32})$$

together with

$$\begin{aligned} \frac{\partial^4 E}{\partial y_2^k \partial y_2^n \partial y_1^m \partial y_1^j} &\rightarrow -u_{kn}^a (i_{ab} u_{mj}^b + i_{cv}^f i_{af} u_m^v u_j^c) - u_{mj}^a i_{cv}^f i_{af} u_k^v u_n^c \\ &+ (S_{abcd} - i^{1f} i_{a1b} i_{dc}^e i_{fe}) u_k^a u_n^b u_m^c u_j^d, \end{aligned}$$

or

$$\frac{\partial^4 E}{\partial y_2^k \partial y_2^n \partial y_1^m \partial y_1^j} \rightarrow -(u_{kn}^a + i_{cv}^a u_k^v u_n^c) i_{ab} (u_{mj}^b + i_{cv}^b u_m^v u_j^c) + S_{abcd} u_k^a u_n^b u_m^c u_j^d. \quad (\text{E.33})$$

In this way we arrive at the reductions

$$\frac{\partial^3 E}{\partial y_2^k \partial y_1^m \partial y_2^n} \rightarrow -\left(-\frac{1}{F^3} (h_{nm} l_k + h_{nm} l_k) + \frac{1}{F^2} C_{kmn}\right) \quad (\text{E.34})$$

and

$$\frac{\partial^3 E}{\partial y_2^k \partial y_1^m \partial y_1^n} \rightarrow -\left(-\frac{1}{F^3} (h_{nk} l_m + h_{mk} l_n) + \frac{1}{F^2} C_{kmn}\right). \quad (\text{E.35})$$

Taking into account (E.13), we can write

$$\frac{\partial^4 E}{\partial y_2^k \partial y_2^n \partial y_1^m \partial y_1^j} \rightarrow -\left(\frac{\partial}{\partial y^n} \frac{1}{F} h_k^t + \frac{1}{F} C_{kn}^t\right) h_{tq} \left(\frac{\partial}{\partial y^j} \frac{1}{F} h_m^q + \frac{1}{F} C_{mj}^q\right) + S_{abcd} u_k^a u_n^b u_m^c u_j^d,$$

which is

$$F^4 \frac{\partial^4 E}{\partial y_2^k \partial y_2^n \partial y_1^m \partial y_1^j} \rightarrow (l_n h_k^t + l_k h_n^t - F C_{kn}^t) (-l_j h_{tm} - l_m h_{tj} + F C_{mj}^t) + F^4 S_{abcd} u_k^a u_n^b u_m^c u_j^d,$$

or

$$\begin{aligned} F^4 \frac{\partial^4 E}{\partial y_2^k \partial y_2^n \partial y_1^m \partial y_1^j} &\rightarrow l_n (-l_j h_{km} - l_m h_{kj} + F C_{mj}^k) + l_k (-l_j h_{nm} - l_m h_{nj} + F C_{mj}^n) \\ &+ F C_{knm} l_j + F C_{knj} l_m - F^2 C_{kn}^t C_{mj}^t + F^4 R_{abcd} u_k^a u_n^b u_m^c u_j^d. \end{aligned} \quad (\text{E.36})$$

Now we differentiate the preservation law (E.2) with respect to y_1^m and y_2^n and make $y_2 \rightarrow y_1$, which yields

$$\begin{aligned} &\partial_i h_{mn} - h_{mn} \frac{1}{F^2} \partial_i F^2 + N_{im}^k h_{kn} + N_{in}^k h_{km} \\ &+ \frac{1}{F} N_{im}^k \left(-(h_{nk} l_m + h_{mn} l_k) + F C_{kmn} \right) + \frac{1}{F} N_{in}^k \left(-(h_{km} l_m + h_{nm} l_k) + F C_{kmn} \right) = -\frac{2}{H} H_i h_{mn}. \end{aligned}$$

On so dong we come to the following sought equality

$$\mathcal{D}_i h_{mn} - \frac{2}{F} h_{mn} d_i F = -\frac{2}{H} H_i h_{mn} \quad (\text{E.37})$$

with

$$\mathcal{D}_i h_{mn} = \partial_i h_{mn} + N^k_i \frac{\partial h_{mn}}{\partial y^k} + N^k_{im} h_{kn} + N^k_{in} h_{km} \quad (\text{E.38})$$

and

$$d_i F = \partial_i F + N^k_i l_k. \quad (\text{E.39})$$

In the vanishing case

$$\partial_i F + N^k_i l_k = 0 \quad (\text{E.40})$$

we obtain by differentiation the equalities

$$\partial_i l_n + \frac{1}{F} N^k_i h_{kn} + N^k_{in} l_k = 0 \quad (\text{E.41})$$

and

$$\partial_i h_{mn} + N^k_i \frac{\partial h_{kn}}{\partial y^m} + N^k_{im} h_{kn} + N^k_{in} h_{km} - \frac{1}{F} N^k_i h_{kn} l_m + \frac{1}{F} N^k_{il_k} h_{mn} + F N^k_{inm} l_k = 0.$$

So we can write

$$\mathcal{D}_i h_{mn} + F N^k_{inm} l_k = 0 \quad (\text{E.42})$$

together with

$$F N^k_{inm} l_k = \frac{2}{H} H_i h_{mn}. \quad (\text{E.43})$$

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